

Assignment Number 3

Problem 1 (6 points) Show the following limits:

a) $\lim_{z \rightarrow \infty} \frac{4z^5}{z^5 - 42z} = 4;$ b) $\lim_{z \rightarrow \infty} \frac{z^4}{z^2 + 42z} = \infty;$

c) $\lim_{z \rightarrow \infty} \frac{(az + b)^3}{(cz + d)^3} = \frac{a^3}{c^3}$ if $a, b, c, d \in \mathbb{C}, c \neq 0.$

Problem 2 (3 points) Show that the following functions are defined on all of \mathbb{C} , but are nowhere analytic (here $z = x + iy$):

a) $z \mapsto 2xy + i(x^2 + y^2)$ b) $z \mapsto e^y e^{ix}.$

Problem 3 (4 points) Determine where the function $z \mapsto x^3 + i(1 - y)^3$ is:

a) analytic; b) differentiable (here $z = x + iy$).

Problem 4 (3 points) For $z = x + iy$, define $f(z) = \sqrt{|xy|}.$

a) Show that f satisfies the Cauchy-Riemann equations at the origin. (Note: you will need to use the definition of the partial derivative to calculate $u_x(0, 0)$ and $u_y(0, 0)$).

b) Show that f is not differentiable at the origin. (Hint: approach on a suitable line).

c) Explain why this doesn't contradict any of the results from class.

Problem 5 (4 points)

(a) Define precisely what it means for a curve in \mathbb{C} to be *rectifiable*.

(b) Give an example of a non-rectifiable curve in \mathbb{C} . You don't have to prove it is non-rectifiable.

Note: you will need to provide at least one reference, properly cited. This is not allowed to be wikipedia.

Problem 6 (2 bonus points) Consider sequences defined as follows:

$$x_0 = 0, \quad x_1 = x_2 = 1, \quad x_3 = x_4 = 1 - \frac{1}{2}, \quad x_5 = x_6 = 1 - \frac{1}{2} + \frac{1}{4}, \dots \quad \text{and}$$
$$y_0 = y_1 = 0, \quad y_2 = y_3 = 1, \quad y_4 = y_5 = 1 - \frac{1}{2}, \quad y_6 = y_7 = 1 - \frac{1}{2} + \frac{1}{4}, \dots$$

Set $z_n = x_n + iy_n$, and define I_n to be the closed line segment (i.e., including end-points) from z_n to z_{n+1} , for $n \in \mathbb{N}_0$. Finally, set:

$$\Omega = \bigcup_{n=0}^{\infty} I_n, \quad \text{and } \Lambda = \Omega \cup \left\{ \frac{2}{3} + \frac{2}{3}i \right\}.$$

a) Show that Λ is connected. (A sketch is not a proof, but could be helpful in fixing ideas.)

b) Show that Λ is *not* piecewise affinely path connected, in the sense defined in class (Lecture 12).

c) Do these results contradict anything from class? Explain.

Due: 2:00PM, Friday, 26/04/2024

Current assignments will be available at

<http://www.maths.uq.edu.au/courses/MATH3401/Tutorials.html>