# Complex Analysis: Supplementary notes I

## 1 FIELDS

The complex numbers, $\mathbb{C}$, form a *field*. This means that we have a *set*, here $\mathbb{C}$, and two binary operations

$$\textit{addition}, + : \mathbb{C} \times \mathbb{C} \to \mathbb{C}, \quad \text{and } \textit{multiplication}, \cdot : \mathbb{C} \times \mathbb{C} \to \mathbb{C},$$

for which the axioms **F1-F7** below hold true. Note that multiplication can also be denoted by $\times$, or simply by juxtaposition.

**(F1) Associativity of addition:** $(r + s) + t = r + (s + t)$ for all $r, s, t \in \mathbb{C}$;

**(F2)(i) Existence of additive identity:** There exists $0 \in \mathbb{C}$ such that

$$r + 0 = r \qquad \text{for all } r \in \mathbb{C};$$

**(F2)(ii) Existence of additive inverse:** Given $r \in \mathbb{C}$ there exists $s \in \mathbb{C}$ such that

$$r + s = 0 \qquad (\text{we write } s = -r);$$

**(F3) Commutativity of addition:** $r + s = s + r$ for all $r, s \in \mathbb{C}$;

**(F4) Associativity of multiplication:** $(rs)t = r(st)$ for all $r, s, t \in \mathbb{C}$;

**(F5)(i) Existence of multiplicative identity:** There exists $1 \in \mathbb{C}$ with $1 \neq 0$ such that

$$r \cdot 1 = r \qquad \text{for all } r \in \mathbb{C};$$

**(F5)(ii) Existence of multiplicative inverse:** Given $r \in \mathbb{C} \setminus \{0\}$ there exists $t \in \mathbb{C}$ such that

$$rt = 1 \qquad (\text{we write } t = r^{-1});$$

**(F6) Commutativity of multiplication:** $rs = sr$ for all $r, s \in \mathbb{C}$;

**(F7) Distributive Law:** $r(s + t) = rs + rt$ for all $r, s, t \in \mathbb{C}$.

# 2 SEQUENCES

**DEFINITION** Let $f : \mathbb{N} \to \mathbb{C}$ (or $\mathbb{N}_0 \to \mathbb{C}$). Then $f$ is called a *sequence*.

If $f(n) = a_n$, then $a_n$ is called the *nth term*. Various notation is used: e.g. $\{a_n\}$ or $(a_n)$ or $\{a_0, a_1, a_2, \ldots\}$.

**DEFINITION** A sequence $\{a_n\}$ is said to *converge* to a limit $a$ if for every $\varepsilon > 0$ there is an integer $N$ such that $|a_n - a| < \varepsilon$ whenever $n \geq N$. (The number $N$ may depend on $\varepsilon$; in particular, smaller $\varepsilon$ may (and often does) require larger $N$). In this case we write:

$$\lim_{n\to\infty} a_n = a$$

or

$$a_n \to a \text{ as } n \to \infty.$$

A sequence that does not converge is said to *diverge*.