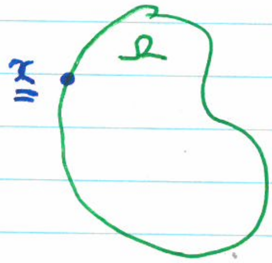
LECTURE 26

§116 (8Ed §115) PHYSICAL PROBLEMS.

Physical configurations often modelled by sol^ns of partial differential equations (PDE)

Generally, interested in solving PDE subject to given initial/boundary conditions.

e.g. (D) $\begin{cases} \Delta u = 0 \text{ in } \Omega \\ u|_{\partial\Omega} = \varphi \quad (*) \end{cases}$



(*) says: $u(\underline{\underline{x}}) = \varphi(\underline{\underline{x}})$ for $x \in \partial\Omega$.

$\varphi : \partial\Omega : \mathbb{R}$; Ω, φ are known/given.

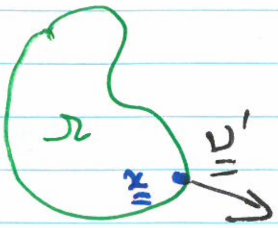
u is unknown.

(D) is called the Dirichlet problem for Laplace's equation, a.k.a. boundary problem of the first kind.

One way to "solve" (D) is to find u that minimises $\int_\Omega |\nabla u|^2 dx$ subject to $u|_{\partial\Omega} = \varphi$.

(rmk: show that $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_\Omega |\nabla(u+\varepsilon\phi)|^2 dx = 0$, where ϕ is any "perturbation" with compact support in Ω). set on which ϕ is not zero.

Also important: boundary cond^{n}s of the second kind, a.k.a. Neumann bdy cond^{n}s.



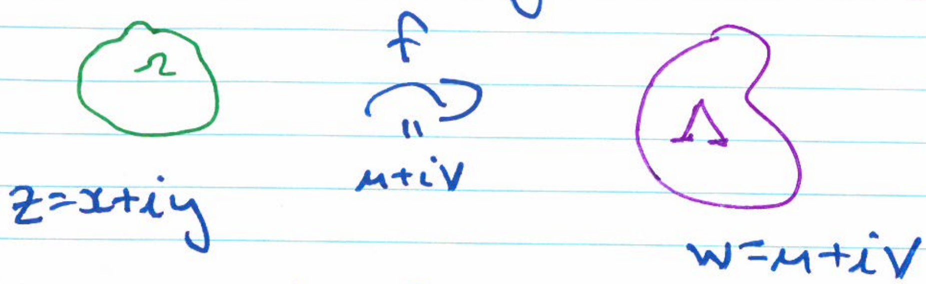
exterior unit normal

$$(N)\quad \begin{cases} \Delta u = 0 \\ \dfrac{\partial u}{\partial \underline{\underline{\nu}}'} = \psi \quad \text{on } \partial\Omega. \end{cases}$$

$$\frac{\partial u}{\partial \underline{\underline{\nu}}'} = \nabla u(\underline{\underline{x}})\cdot \nu'(\underline{\underline{x}}).$$

In practice, often have homogeneous Neumann b.c., i.e. $\psi = 0$ (no-slip conditions).

Transformation of harmonic functions.



Thm: if f is conformal & h is harmonic in Δ, then H is harmonic in Ω, where

$$H(x,y) = h(u(x,y), v(x,y)).$$

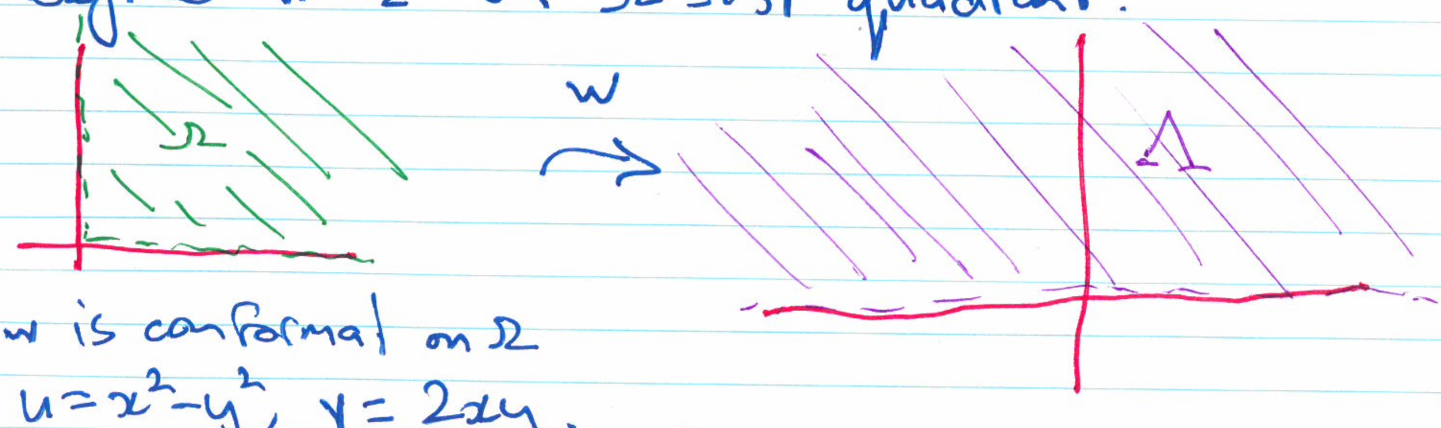
Pf: messy in general, but relatively straightforward if Δ is simply connected (§115, 8th ed §104)

E.g. $h(u,v) = e^{-v}\sin u$ is harmonic in the UHP (upper half plane).

To see that, note that e^{-v} is C^∞, as is $\sin u$ $\Rightarrow$ h is C^∞: in particular, h & all partials exist & are cts on $\mathbb{R}^2$.

$$\left.\begin{array}{ll} h_u = e^{-v}\cos u & h_{uu} = -e^{-v}\sin u \\ h_v = -e^{-v}\sin u & h_{vv} = e^{-v}\sin u \end{array}\right\} \Rightarrow \Delta h = 0.$$

Define $w = z^2$ on Ω = 1st quadrant.



w is conformal on Ω

$u = x^2 - y^2$, $v = 2xy$.

Thm $\Rightarrow$ $H(x,y) = e^{-2xy}\sin(x^2-y^2)$ is harmonic on Ω.

Add in bdy cond^n's as well!

$z = x + iy$ $\qquad\qquad\qquad\qquad$ $w = u + iv$.

f conformal

C smooth (C^∞) curve in Ω

(can take C in $\partial\Omega$ with a bit more care).

$H(x,y) = h(u(x,y), v(x,y))$.

(1) Dirichlet b.c. on Γ, i.e.,

$h(u,v) = \varphi$ on Γ : then

$$\boxed{H(x,y) = \varphi \text{ on } C}.$$

(2) If we have homogeneous Neumann b.c. on Γ, i.e., $\frac{\partial h}{\partial \underline{n}} = 0$ for $\underline{n}$ normal to Γ, then

$$\boxed{\frac{\partial H}{\partial \underline{N}} = 0 \text{ for } \underline{N} \text{ normal to } C}.$$