LECTURE 25

Harmonic functions are sol^{n}s of Laplace's eq^n:

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} = 0 .$$

Examples of harmonic f^{n}s:

* On $\mathbb{R}^2 \setminus \{(0,0)\}$:

$$f(x,y) = \ln(\|(x,y)\|) = \ln\left(\sqrt{x^2+y^2}\right) \text{ is harmonic}$$

* On $\mathbb{R}^n \setminus \{\underline{0}\}$ $n \geq 3$

$$f(\underline{x}) = 1/\|x\|^{n-2} = 1/(x_1^2+\cdots+x_n^2)^{n-2/2} \text{ is harmonic}$$

* On $\mathbb{R}$: $f(x) = ax+b$ $a,b \in \mathbb{R}$.

Look at $\mathbb{R}^2$.

Precise def^n of a harmonic f^n:

A f^n $u : \Omega \to \mathbb{R}$ ($\Omega \subset \mathbb{R}^2$) is harmonic if:

* u & its derivatives of first order (i.e., u_x, u_y) exist & are cts on Ω;
* u_{xx} & u_{yy} are cts on Ω; &
* $u_{xx} + u_{yy} = 0$, i.e., $\Delta u = 0$ in Ω.

THM: If $\underset{x+iy}{f(z)} = u(x,y) + i\,v(x,y)$ is analytic in $\Omega \subseteq \mathbb{C}$, then u & v are harmonic.

Pf: recall §55: f is analytic $\Rightarrow$ u & v have cts partials of all orders & C/R hold in Ω, i.e.,

$$u_x = v_y \quad \& \quad u_y = -v_x. \qquad (1)$$

∂_x of (1) $\Rightarrow$ $u_{xx} = v_{yx}$ & $u_{yx} = -v_{xx}$ (2)

∂_y of (1) $\Rightarrow$ $u_{xy} = v_{yy}$ & $u_{yy} = -v_{xy}$ (3)

Since all partials of all orders are cts,

$u_{xy} = u_{yx}$ & $v_{xy} = v_{yx}$ (Clairaut's th^{m}.)

So (2) & (3) $\Rightarrow$ $\Delta u = 0$ & $\Delta v = 0$ □

Def^{n}: If u & v are harmonic & satisfy C/R, then v is called a harmonic conjugate of u.

"not "the"

THM: $f = u + iv$ is analytic in $\Omega \Leftrightarrow$ v is a harmonic conjugate of u.

Pf: ($\Rightarrow$) done in previous th^{m}.

($\Leftarrow$) v is a harm conj of u says u & v are both harmonic, so v, u, u_x, u_y, v_x, v_y exist & are cts; & C/R hold throughout Ω $\Rightarrow$ $f = u + iv$ is analytic in Ω. □

Finding harmonic conjugates.

Firstly, note:

Suppose v & w are harmonic conjugates of u.

$\Rightarrow$ $u+iw$ & $u+iv$ are both analytic.

$C/R_I \Rightarrow u_x = \underbrace{v_y = w_y}_{*}$.

Integrate $*$ w.r.t. y $\Rightarrow v = w + \Phi(x)$. (4)

$C/R_{II} \Rightarrow u_y = \underbrace{-v_x = -w_x}_{☺}$.

Integrate ☺ w.r.t. x $\Rightarrow v = w + \Psi(y)$ (5)

Compare (4) & (5) $\Rightarrow \Phi(x) = \Psi(y)$, which hence must be a constant $\in \mathbb{R}$, i.e.,

$$\boxed{v = w + c}$$

Use a similar procedure to find the harm. conj. of a given harm. f^n u.

E.g., $u(x,y) = y^3 - 3x^2 y$ on $\mathbb{R}^2$: find a harm. conj. of u.

<u>Sol^n</u>: u is a polynomial f^n of x & y, so has cts partials of all orders.

Further $u_{xx} + u_{yy} = 0$ ✱.

Let v be a harm. conj of u.

$C/R_{I} \Rightarrow u_x = v_y$,

i.e., $v_y = -6xy$.

Integrate w.r.t. $y \Rightarrow v = -3xy^2 + \phi(x)$. 6

$C/R_{II} \Rightarrow u_y = -v_x$.

$$3\cancel{y^2} - 3x^2 = 3\cancel{y^2} - \phi'(x).$$

$$\Rightarrow \phi'(x) = 3x^2$$

$$\Rightarrow \phi(x) = x^3 + c \qquad c \in \mathbb{R}.$$

So $(c=0) \Rightarrow v = -3xy^2 + x^3$ is a harm. conj. of u.

Note: for $f(z) = iz^3$, $u = \mathrm{Re}(f)$, $v = \mathrm{Im}(f)$.

RMK: v is a harmonic conjugate of u

$\not\Rightarrow$ u is a harmonic conjugate of v.

E.g., $u = x^2 - y^2$, $v = 2xy$.

$\Rightarrow u + iv = (x+iy)^2 = z^2$, which is entire

$\Rightarrow v$ is a harmonic conjugate of u.

BUT, if u were a harmonic conjugate of v,
then $g = v + iu$ would be analytic.
Check via C/R: g is nowhere analytic ✫.

RMK: Suppose u is harmonic on a simply connected domain Ω.

Then (§115 8Ed §104)

$$\boxed{u \text{ has a harm conj. on } \Omega.}$$

next physical problems.