LECTURE 24

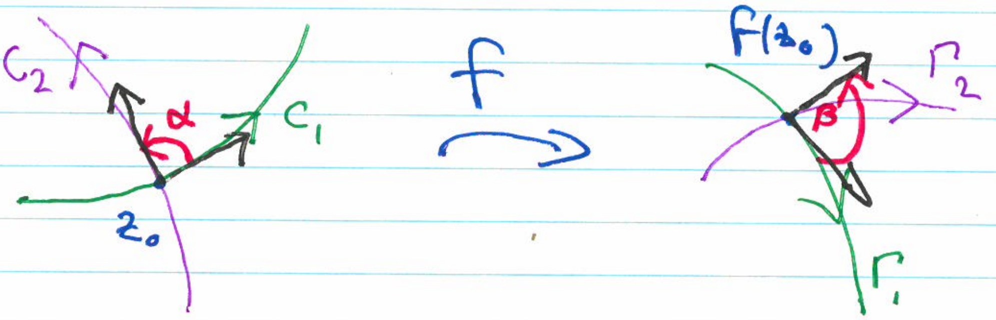
§112 (8Ed §101) Conformal maps.

* $f: z \mapsto w$
* f analytic
* $f'(z_0) \neq 0$.

Then locally (near z_0), f preserves

* angle;
* orientation;
* shape.

f is called <u>conformal</u> at z_0.

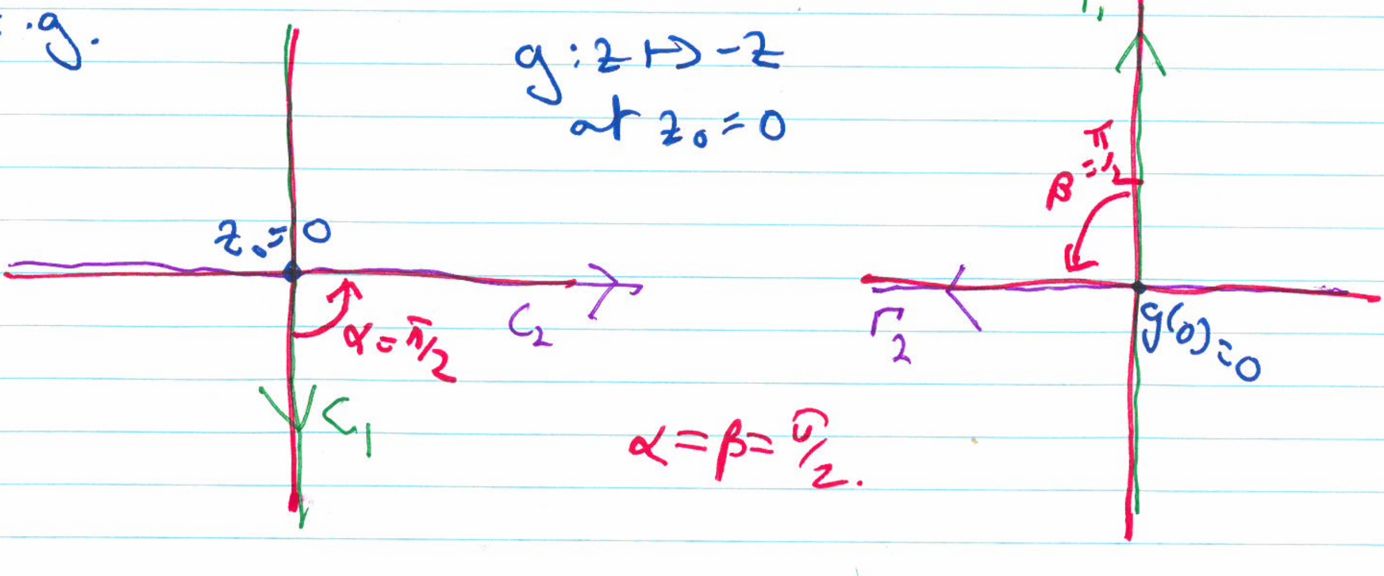


$\Gamma_1 = f(C_1)$, $\Gamma_2 = f(C_2)$

$$\boxed{\text{Conformality} \Rightarrow \beta = \alpha}$$

E.g.

$g: z \mapsto -z$
at $z_0 = 0$



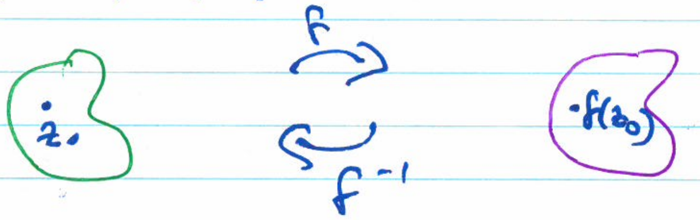
$\alpha = \beta = \pi/2$.

analytic f^n with $f'(z_0) = 0$: z_0 is a critical point of f. Angle will not be preserved.
Can show: angle will be multiplied by m, where m is the smallest integer s.t.

$$\boxed{f^{(m)}(z_0) \neq 0}$$

RMK If orientation is not necessarily preserved, but angle magnitude is, the map is called isogonal.

Conformality $\Rightarrow$ locally 1-1 & onto, i.e., f has a local inverse:



Inverse f^n Th^m (MATH2400/2401) in $\mathbb{R}^2$ says $f: (x,y) \mapsto (u,v)$ sufficiently smooth is invertible if

$$\det(J_f) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \neq 0.$$

Jacobian

In our case, we know f is analytic, so u_x, u_y, v_x, v_y are all cts in a nbhd of $x_0 + iy_0 = z_0$, and

and $\det(J_f) = u_x v_y - u_y v_x$

$\qquad v_y = u_x, \quad u_y = -v_x$ via C/R

$= u_x^2 + v_x^2$

$= |u_x + i v_x|^2$

$= |f'|^2 \neq 0$ at z_0 $\square$.

Harmonic f^n's in $\mathbb{R}^n$.

$\Omega \subseteq \mathbb{R}^n$: look for $U : \Omega \to \mathbb{R}$ s.t.

$\Delta U = 0$

Laplacian or Laplace operator

(sometimes ∇^2)

where $\Delta U = \sum_{j=1}^{n} U_{jj} = \sum_{j=1}^{n} \frac{\partial^2 U}{\partial x_j^2}$

in $\mathbb{R}^2$: $\Delta U = U_{xx} + U_{yy}$

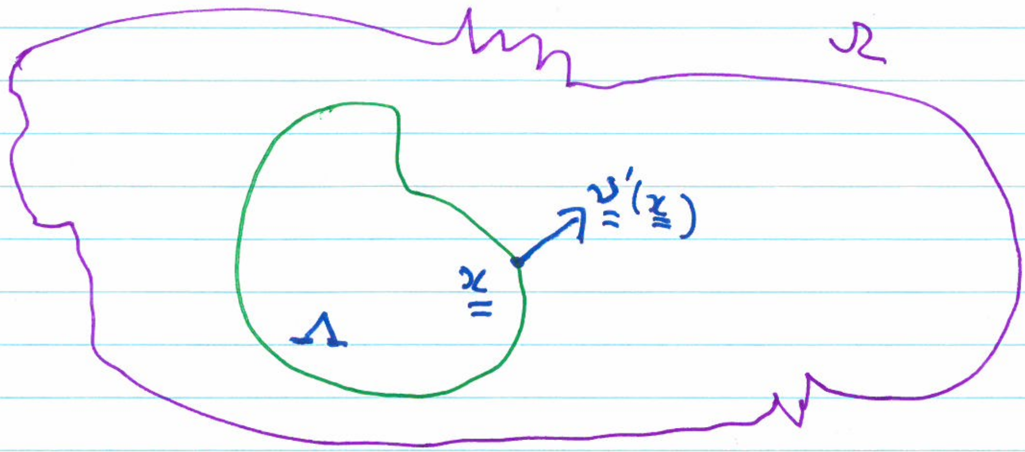
in $\mathbb{R}^3$: $\Delta U = U_{xx} + U_{yy} + U_{zz}$

Models many physical situations in "steady state".

Motivation: $\Omega \subseteq \mathbb{R}^2$ or $\mathbb{R}^3$.

Λ: "suff. smooth subdomain of Ω" with an exterior normal $\underline{\underline{\nu}}(\underline{\underline{x}})$ on $\partial\Lambda$, & external unit normal $\underline{\underline{\nu}}'(\underline{\underline{x}})$.

"$=$" denotes a vector quantity.



U = density of something "in equilibrium" in Ω

$\underline{\underline{F}}$ = flux density of U "in equilibrium" in Ω.

$$\int_{\partial\Lambda} \underline{\underline{F}} \cdot \underline{\underline{\nu}}' \, dS' = 0$$ dS' = surface measure on $\partial\Lambda$.

Gauß $\Rightarrow \int_{\Lambda} \operatorname{div} \underline{\underline{F}} \, d\underline{\underline{x}} = 0$.

$d\underline{\underline{x}} = dx\,dy$ in 2D

$d\underline{\underline{x}} = dx\,dy\,dz$ in 3D.

Since A is (essentially) arbitrary, there holds:

$$\operatorname{div} \underline{\underline{F}} = 0 \text{ in } \Omega,$$

i.e.,

$$\sum_{j=1}^{n} \partial_j \underline{\underline{F}} = 0 \text{ in } \Omega \quad (*)$$

In many physical situations,

$\underline{\underline{F}} = c \nabla \bar{u}$, with c typically negative.

$$(*) \Rightarrow c(\operatorname{div} \nabla \bar{u}) = 0$$

i.e. $\Delta \bar{u} = 0$.

Can also study $\dfrac{\partial \bar{u}}{\partial t} = \alpha \Delta \bar{u} \quad (*)$

- If $\bar{u}$ is the concentration of a chemical, then $(*)$ is Fick's law of chemical diffusion;
- If $\bar{u}$ is temperature, $(*)$ is Fourier Law of heat conduction;
- If $\bar{u}$ is the electric potential, $(*)$ is Ohm's Law of electrical conduction.

Office hours: 1030–1130 tomorrow, now.