LECTURE 23

Let f be analytic in & on C, a simple closed contour in $\mathbb{C}$, traversed in the +ve sense, & let $z_0 \in \text{Int}\, C$.

Cauchy (Lec. 22) $\Rightarrow$ $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz$.

There further holds:

Thm §55 (8 Ed §52)

$$\boxed{f^{(n)}(z_0)} = \frac{n!}{2\pi i}\int \frac{f(z)}{(z-z_0)^{n+1}}\,dz, \qquad (1)$$

for $n \geq 1$

nth derivative of f at z_0

works for $n=0$, $0! = 1$ $f^{(0)} = f$.

Pf: see ex 9 §57 (8Ed Ex 9 §52)

THM (*) If f is analytic at z_0, then its derivatives of all orders exist & are analytic at z_0.

Pf: f is analytic at $z_0 \Rightarrow f$ is analytic on $B_\varepsilon(z_0)$ for some $\varepsilon > 0$. Then for

$C = \{z_0 + \frac{\varepsilon}{2}e^{i\theta}, 0 \leq \theta \leq 2\pi\}$, f is analytic on C & in Int C, so

$$(1) \Rightarrow f''(z) = \frac{1}{\pi i}\int_C \frac{f(\xi)}{(\xi - z)^3}\,d\xi \quad \forall z \in \text{Int}\, C.$$

$\Rightarrow f''$ exist on Int C $\Rightarrow f'$ is analytic on Int C, & at z_0 in particular.

Reapply to f' to show f'' is analytic, etc. □

cf the situation in $\mathbb{R}$, e.g. $f(x)=|x|^3$: f, f', f'' are all cts on $\mathbb{R}$, but $f'''(0)$ doesn't exist.

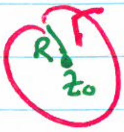
<u>RMK</u> For $f=u+iv$: f is analytic at $z_0=x_0+iy_0$ $\Rightarrow$ partials of all orders of u & v exist & are cts at (x_0,y_0).

<u>THM</u> (Morera) Let f be cts on a domain $\Omega \subseteq \mathbb{C}$. If $\int_C f = 0$ $\forall$ closed contours lying in Ω, then f is analytic in Ω.

<u>PF:</u> By Thm §48 (Lec 21), f has a primitive on Ω, say F. But then $F'=f$ exists & is cts on Ω by conditions of the thm, so F is analytic. So by Thm (*), f $(=F')$ is also analytic on Ω. $\square$

A number of nice results follow from Th^{m} §55.

I. Let f be analytic in & on $C_R(z_0) = \{z_0 + Re^{i\theta},\ 0 \leq \theta \leq 2\pi\}$.



Then: $$\boxed{|f^{(n)}(z_0)| \leq \frac{n!\, M_R}{R^n} \quad (2)}$$

where $M_R = \max_{z \in C_R} |f(z)|$.

Pf: note that M_R is well defined (& in particular, finite) by Extreme Value Th^{m}.

$$|f^{(n)}(z_0)| \overset{\text{Th}^{m}\ \S 55}{=} \left| \frac{n!}{2\pi i} \int_{C_R} \frac{f(z)}{(z-z_0)^{n+1}}\, dz \right|$$

$$\leq \frac{n!}{2\pi} \int_{C_R} \frac{\underbrace{|f(z)|}_{\leq M_R \text{ on } C_R}}{\underbrace{|z-z_0|^{n+1}}_{= R^{n+1} \text{ on } C_R}}\, dz$$

$$\leq \frac{n!\, M_R}{2\pi R^{n+1}} \int_{C_R} dz$$

$$= \frac{n!\, M_R}{\cancel{2\pi R}\, R^n}\, \cancel{2\pi R}$$

$$= \frac{n!\, M_R}{R^n}$$

$\square$

Rmk: leads to e.g. Bieberbach conjecture.

II : Liouville's Thm:

If $f: \mathbb{C} \to \mathbb{C}$ is bded & entire, then f is constant.

Pf: Suppose that $|f| < M$ on $\mathbb{C}$, f is entire, $z_0 \in \mathbb{C}$.

Apply (2) for $n=1$ on $C_R = \{z_0 + Re^{i\theta}, 0 \leq \theta \leq 2\pi\}$

$$|f'(z_0)| \leq \frac{1!\,M}{R^1} = M/R \quad (**)$$

Letting $R \to \infty$ & noting the LHS of (**) is independent of R, we see there must hold $f'(z_0) = 0$. Since z_0 was arbitrary, f must be constant.

IMPORTANT entire is crucial here.

$\exists$ nonconstant, bded analytic f^{n}s on "large" (unbounded) domains in $\mathbb{C}$.

E.g on the UHP (upper half plane) $\{z : \operatorname{Im} z > 0\}$, consider $z \mapsto e^{iz}$.

For $z = x + iy \in$ UHP, $y > 0$, so

$$|e^{iz}| = |e^{i(x+iy)}| = |e^{ix} e^{-y}| = e^{-y} < 1$$

So e^{iz} is bded on UHP.

III. Fund. th^m of algebra. (Tutes wk 9).

Next: §11.2 conformal maps.