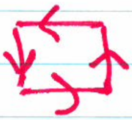
LECTURE 22

§50 (8 Ed §46) Cauchy–Goursat

Let C be a simple closed contour in $\mathbb{C}$. If f is analytic on C & its interior, then

$$\boxed{\int_C f(z)\,dz = 0}$$

Rmk: $\int_C f(z)\,dz = 0 \not\Rightarrow f$ is analytic in & on C: consider e.g. $\int_C z^n\,dz = 0$, $n = -2, -3, -4, \ldots$ for C any circle centred at 0.

Pf: ① Do it for  ;

② Approximate.

A key step in the proof of Cauchy-Goursat is the <u>M-ℓ estimate</u>:

Suppose f is cts on a contour C given by $z(t)$, $a \le t \le b$.

Then $\exists M$ s.t. $|f(z)| \le M \ \forall z \in C$.

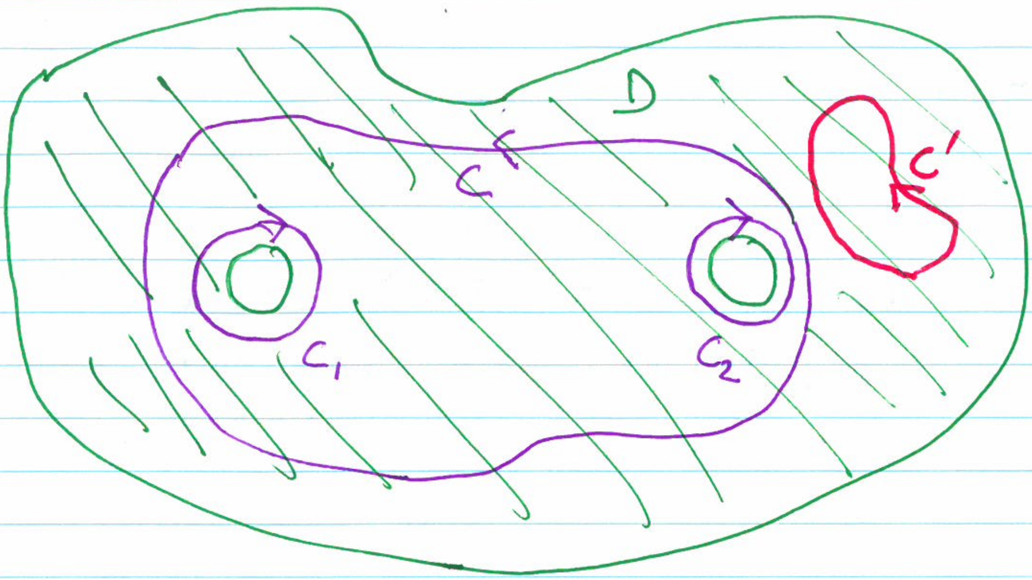
($f \circ z : [a,b] \to \mathbb{C}$ is cts; $|f \circ z| : [a,b] \to \mathbb{R}$ is cts, so the extreme value thm in $\mathbb{R}$ from MATH2400/2401 $\Rightarrow$ result.).

$$\text{So, } \left|\int_C f(z)\,dz\right| = \left|\int_a^b f(z(t))\, z'(t)\,dt\right|$$
$$\le \int_a^b |f(z(t))| \cdot |z'(t)|\,dt$$
$$\le M \int_a^b |z'(t)|\,dt$$
$$= M\ell$$

where $\ell = \ell(C) =$ length of C.

This is the M-ℓ estimate: $\boxed{\int_C f(z)\,dz \le M\ell.}$

<u>Remark</u>: a domain D is simply connected if, for every simple closed contour C in D, there holds $\operatorname{Int} C \subset D$, i.e., "no holes". (i.e., all simple closed contours in D are null homotopic in D).



C' is null homotopic in D, & C is not.

Cauchy-Goursat extension to multiply-connected domains.

Let f is analytic on simple closed contours C & $C_1, C_2, \dots, C_n \subset \operatorname{Int} C$, $C_1, \dots, C_n$ disjoint, & f is analytic on the interior of the domain bounded by $C, C_1, \dots, C_n$.

Let C be truly oriented & $C_1, \dots, C_n$ be negatively oriented, orientations taken in $\mathbb{C}$.

Then: $\int_C f + \sum_{j=1}^{n} \int_{C_j} f = 0$.

§54 (8Ed §50)
Cauchy Integral Formula

Let f be analytic on & inside a simple closed curve C that is positively oriented. (i.e., if $z(t)$ parametrises C, then as $t \nearrow$, Int C stays on the curve's LHS).

Then if $z_0 \in \text{Int}\, C$, we have

$$\boxed{\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz \quad (1) \quad \text{i.e.,} \\ 2\pi i f(z_0) &= \int_C \frac{f(z)}{z-z_0}\,dz. \quad (1)' \end{aligned}}$$

Pf: set $C_\rho = \{z(\theta) = z_0 + \rho e^{i\theta},\ 0 \le \theta \le 2\pi\}$

for $\rho > 0$ suff small that $\text{Int}(C_\rho) \subset D = \text{Int}\, C$.

Then $z \mapsto \frac{f(z)}{z-z_0}$ is analytic on $\text{Int}\, C - \text{Int}\, C_\rho$ & on C & C_ρ.

So C-G extension $\Rightarrow$

$$\int_C \frac{f(z)}{z-z_0}\,dz = \int_{C_\rho} \frac{f(z)}{z-z_0}\,dz.$$

$\Rightarrow$

$$\int_c \frac{f(z)}{z-z_0} - f(z_0)\overset{(*)}{\int_{C_\rho}\frac{dz}{z-z_0}} = \int_{C_\rho}\frac{f(z)-f(z_0)}{z-z_0}\,dz \quad (2)$$

(*) $= 2\pi i$ (cf lec. 21) $\forall \rho > 0$.

Since f is analytic at z_0, it is cts at z_0

$\Rightarrow$ Given $\varepsilon > 0$ $\exists \delta > 0$ s.t.

$$|f(z) - f(z_0)| < \varepsilon \quad \forall |z-z_0| < \delta \qquad (3)$$

Choose $\rho < \delta$.

$$|f(z_0 + \rho e^{i\theta}) - f(z_0)| < \varepsilon \qquad ☺$$

So $|(2)| \Rightarrow$

$$\left|\int \frac{f(z)}{z-z_0} - 2\pi i f(z_0)\right| \leq \int_{C_\rho} \frac{|f(z)-f(z_0)|}{|z-z_0|}\,dz$$

$$= \frac{1}{\rho}\int_{C_\rho} |f(z)-f(z_0)|\,dz$$

$$< \frac{1}{\rho}\,\varepsilon \cdot 2\pi\rho \quad \text{via ☺, M-}\ell$$

$$= 2\pi\varepsilon.$$

Send $\varepsilon \searrow 0 \Rightarrow$ (1)'.