LECTURE 19

A contour is an arc / curve / simple closed curve s.t.

(I) z is cts;

(II) piecewise differentiable, i.e., z' is piecewise cts.

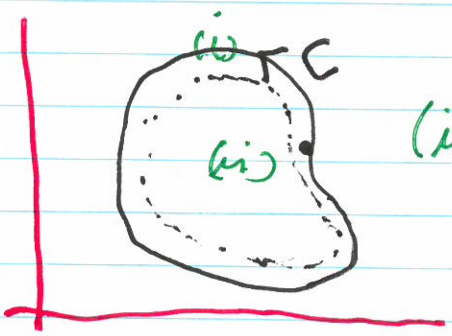
If initial & final values of z coincide, & there are no other self intersections, we have a simple closed contour.

Jordan curve th^{m}: Any simple closed contour divides $\mathbb{C}$ into 3 disjoint sets:

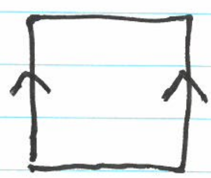
(i) on the curve;

(ii) inside the curve;

(iii) outside the curve.

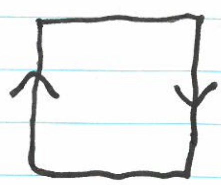


cylinder



RMK: Statement still holds if we remove assumption (II): CARE!

Möbius strip



Contour Integral

$\int_C f(z)\,dz$ or $\int_{z_1}^{z_2} f(z)$, the latter being acceptable if:

a) we know the integral is independent of the path from z_1 to z_2; or

b) if the path is understood.

Suppose the contour C is specified by $z(t)$, $z_1 = z(a)$ & $z_2 = z(b)$, $a \le t \le b$, & suppose f is piecewise cts (pwc) on C. Then:

$$\boxed{\int_C f(z)\,dz = \int_a^b f(z(t))\, z'(t)\, dt.}$$

(cf. line integrals).

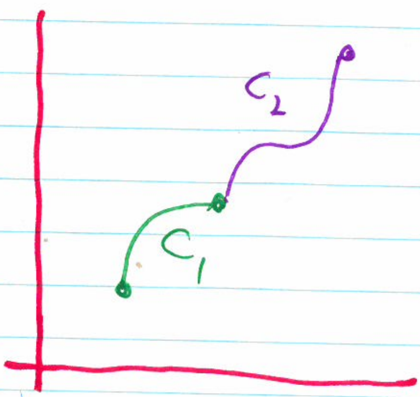
Properties:

* linearity $\int_C (\alpha f)(z)\,dz = \alpha \int_C f(z)\,dz$; $\alpha \in \mathbb{C}$

  $\int_C (f+g)(z)\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$.

* independent of parametrisation (cf end of Lec 18).

$C_1 + C_2$ defines a contour when the end pt of C_1 is the start pt. of C_2.



Given a contour C_1, we define the contour $-C_1$ as follows:

$$w(t) = z(-t) \qquad -b \le t \le -a.$$

Then (check with change of parameter formula from Lec 18):

$$\int_{-C} f(z)\,dz = -\int_{C_1} f(z)\,dz .$$

Hence $C_1 - C_2$ is defined when

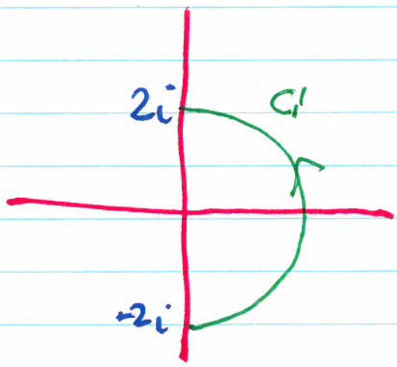
$$\begin{gathered} C_1 - C_2 \\ \| \\ C_1 + (-C_2) \end{gathered}$$

start pt of $-C_2$ = end pt. of C_1.

$\|$

end pt of C_2.

<u>Ex 1</u>: evaluate $I = \int_C \bar{z}\,dz$



$C_1 : z = 2e^{i\theta}$

$-\pi/2 \le \theta \le \pi/2$

C_1 is p.w.c. w.r.t. this parametrisation, & f is cts on C_1 (cts on $\mathbb{C}$).

Note $z'(\theta) = 2ie^{i\theta}$

$$I = \int_{-\pi/2}^{\pi/2} f(z(\theta))\, z'(\theta)\, d\theta = \int_{-\pi/2}^{\pi/2} \overline{(2e^{i\theta})} \cdot 2ie^{i\theta}\, d\theta$$

$$= 4i\int_{-\pi/2}^{\pi/2} \underbrace{e^{-i\theta} e^{i\theta}}_{1}\, d\theta = 4\pi i \quad (*)$$

On C, note $|z| = 2 \Rightarrow z\bar{z} = 4 \Rightarrow \bar{z} = 4/z$.

So (*) $\boxed{\int_C \frac{dz}{z} = \pi i}$

See §45 (8 Ed §41) for more examples, also tute sheets.

Next: antidifferentiation.