# LECTURE 17

## §41–43 (8 Ed §37–39) Integration

Consider a $\mathbb{C}$-valued f$^n$ of a real variable

$w(t) = u(t) + iv(t)$

$t \in \mathbb{R}$

Define $w'(t) = u'(t) + iv'(t)$.

Standard differentiation laws for f$^n$s of a $\mathbb{R}$-variable apply:

* $(cw)' = cw'$ $\quad c \in \mathbb{C}$
* $(w_1 \pm w_2)' = w_1' \pm w_2'$
* $\frac{d}{dt}(e^{ct}) = ce^{ct}$ etc.
* product rule, quotient rule etc.

Definite & indefinite integrals of such f$^n$s:

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt \quad (0) \quad a,b \in \mathbb{R}$$

$$\Rightarrow \operatorname{Re}\left(\int_a^b w(t)\,dt\right) = \int_a^b \operatorname{Re}(w(t))\,dt \quad \& \quad (1)$$

$$\operatorname{Im}\left(\int_a^b w(t)\,dt\right) = \int_a^b \operatorname{Im}(w(t))\,dt \quad (2)$$

$\int_a^\infty w(t)\,dt$ etc defined analogously.

(0) certainly makes sense for cts $w$, i.e., $w \in C^0([a,b])$.

Indeed, ok for so-called piecewise cts f^ns on $[a,b]$, i.e.,

~~$\forall$~~ $u, v : [a,b] \to \mathbb{R}$:

$\exists\, c_1 < c_2 < \dots < c_n \in (a,b)$ s.t. :

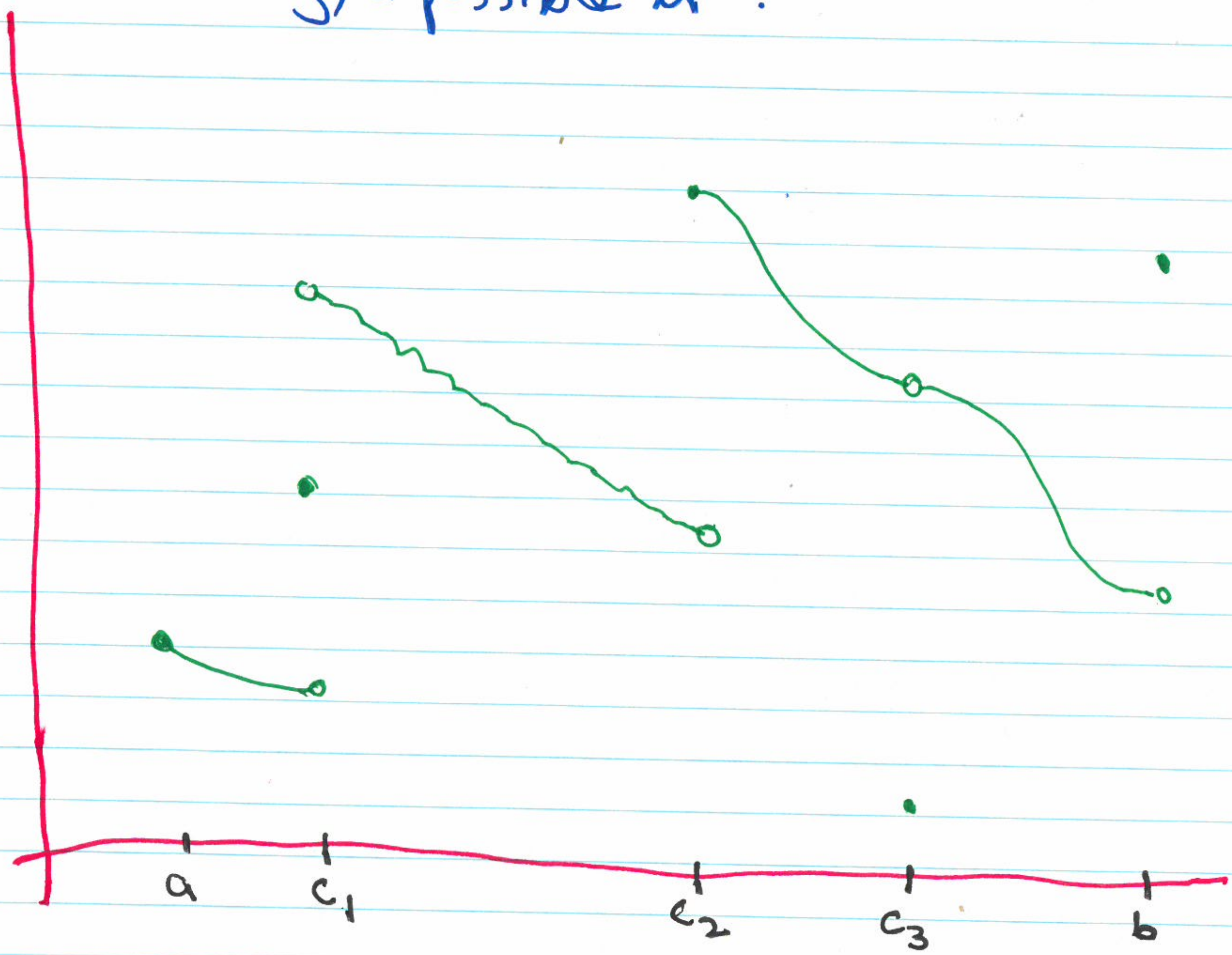
(i) $w$ is cts on $(a,c_1), (c_1,c_2), (c_2,c_3), \dots, (c_{n-1},c_n), (c_n,b)$

(ii) $\lim_{t \to c_j^-} w(t)$, $\lim_{t \to c_j^+} w(t)$ both exist (they may or may not coincide);

(iii) $\lim_{t \to a^+} w(t)$, $\lim_{t \to b^-} w(t)$ both exist.

Here, "exist" means "exist for $u$ & $v$".

e.g., a possible $u$.



Suppose $W(t) = U(t) + iV(t)$ s.t.

$$W' = w \quad \text{on } [a,b]$$

Then the fundamental theorem of calculus holds, in the form

$$\int_a^b w(t)\,dt = W(b) - W(a)$$

The following estimate is crucial.

Suppose $w = u + iv$ is pwc (piecewise cts) on $[a,b]$. Then:

$$\boxed{\left|\int_a^b w(t)\,dt\right| \leq \int_a^b |w(t)|\,dt \quad (3)}$$

PF: If $\int_a^b w(t)\,dt = 0$, LHS of (3) $= 0$, RHS of (3) $\geq 0$, so done.

Otherwise: $\exists\, r > 0$ & $\theta_0 \in \mathbb{R}$ s.t.

$$\int_a^b w(t)\,dt = re^{i\theta_0} \quad (3)'$$

$$\Rightarrow \left|\int_a^b w(t)\,dt\right| = r \quad (4)$$

$(3)' \cdot e^{-i\theta_0} \Rightarrow r = \int_a^b e^{-i\theta_0} w(t)\,dt.$

$$= \mathrm{Re}\left(\int_a^b e^{-i\theta_0} w(t)\,dt\right) \quad \text{since } r \in \mathbb{R}$$

$$= \int_a^b \mathrm{Re}(e^{-i\theta_0} w(t))\,dt \quad (5)$$

$\mathrm{Re}(e^{-i\theta_0} w(t)) \leq |e^{-i\theta_0} w(t)| \leq |w(t)|$

So via (4) & (5): $\left|\int_a^b w(t)\,dt\right| \leq \int_a^b |w(t)|\,dt$,

showing (3) $\square$

Roughly speaking (will be made precise later): 

A <u>contour</u> is a parametrised curve in $\mathbb{C}$.

Given $x(t), y(t)$ cts $[a,b] \to \mathbb{R}$,

$$z(t) = x(t) + iy(t) \quad a \leq t \leq b \text{ defines an}$$

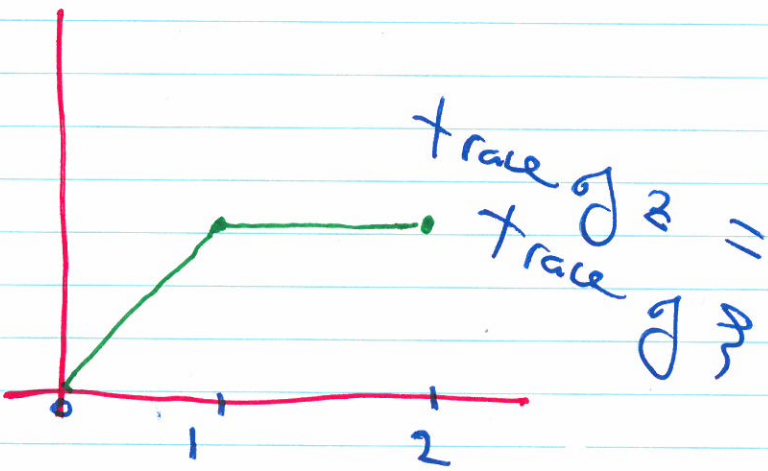
<u>arc</u>.

This is both a set of points in $\mathbb{C}$ (namely the image $z([a,b])$, called the <u>trace</u> of the arc, and also a recipe for parametrising it.

(cf: B-C p.120 (8Ed p.122), which is a bit sloppy).

<u>Ex 1</u>

$$z(t) = \begin{cases} t + it & 0 \leq t \leq 1 \\ t + i & 1 < t \leq 2. \end{cases}$$

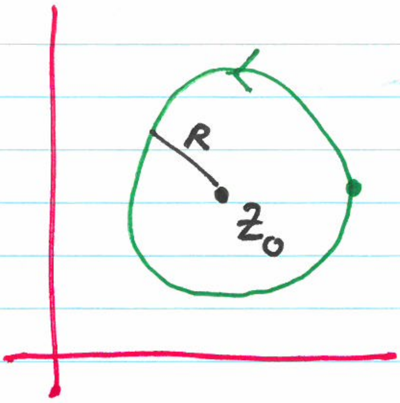



$$\zeta(t) = \begin{cases} 2t + 2it & 0 \leq t \leq \tfrac{1}{2} \\ 1 + i & \tfrac{1}{2} < t \leq 1 \\ t + i & 1 < t \leq 2 \end{cases}$$

Ex2

$z = z_0 + Re^{i\theta}$

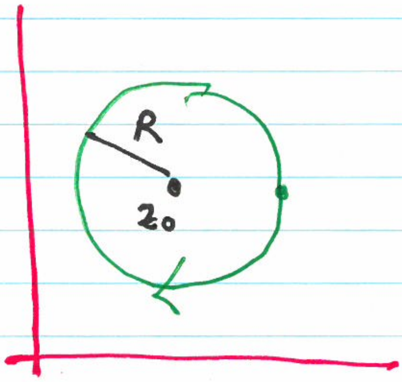
$0 \leq \theta \leq 2\pi$




Ex3

$z = z_0 + Re^{-i\theta}$

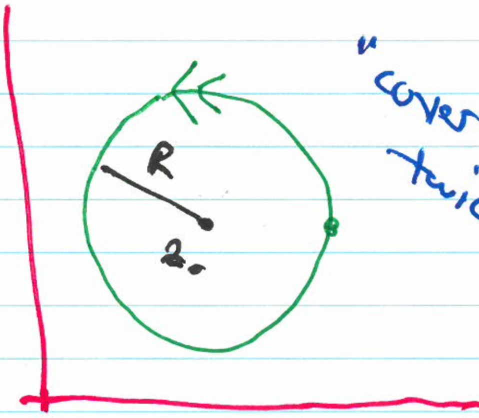
$0 \leq \theta \leq 2\pi$




Ex4

$z = z_0 + Re^{2i\theta}$

$0 \leq \theta \leq 2\pi$




"cover twice"