LECTURE 16

Final rmk on analytic f^n's: if a f^n is differentiable at precisely one point, it is not analytic there or anywhere, e.g. $z \mapsto |z|^2$.

Further on derivatives

$$\frac{d}{dz}(\log z) \qquad : |z| > 0$$

(maybe "$\frac{d}{dz}\log z$" would be a better starting point, as log is multivalued).

Recall $\log z = \ln|z| + i \arg z$

$\quad \overset{"}{re^{i\theta}} \quad = \ln r + i\theta$

$\Rightarrow u = \ln r, \quad v = \theta$

$\Rightarrow u_r = 1/r, \ u_\theta = 0, \ v_r = 0, \ v_\theta = 1$

C/R in polar coords: $\begin{cases} * \ r u_r = v_\theta & \checkmark \\ * \ u_\theta = -r v_r & \checkmark \end{cases}$

So, sufficient conditions for complex differentiability are satisfied on any subset of $\mathbb{C}_*$ s.t.

$\alpha < \theta < \alpha + 2\pi \qquad \alpha$ fixed in $\mathbb{R}$.

On such a set, log (or more precisely, the single-valued f^n we obtain from log) is diff^{ble},

& Lec 15 $\Rightarrow \frac{d}{dz}(\log z) = e^{-i\theta}(u_r + i v_r)$

$$= e^{-i\theta}\left(\frac{1}{r} + 0\right) = \frac{1}{re^{i\theta}} = 1/z.$$

E.g. $\frac{d}{dz} \operatorname{Log} z = 1/z$ for $-\pi < \operatorname{Arg} z < \pi$, $|z| > 0$.

RMK cf. p.94 (8 Ed p.96), cf. Lec 9:

Log is a branch of log.

* for $f(z) = z^c$ $c \in \mathbb{C}_*$ fixed, defined on $\mathbb{C}_*$:

$$f(z) = \exp(c \log z) \ \&$$
$$f'(z) = \exp(c \log z) \cdot c/z \quad (*)$$
$$= z^c \cdot c/z = c z^{c-1} \quad (**)$$

(**) is valid on any domain o.t.f. $\{z : |z| > 0, \ \alpha < \arg z < \alpha + 2\pi\}$, due to the need to choose a branch of log in (*)

RMK: Try for $g(z) = c^z$. ☆

Notation from real analysis

$\Omega \subseteq \mathbb{R}^n$ $\qquad n \geq 1$.

* $C(\Omega) = C^0(\Omega) = \{$cts f^{n}s $: \Omega \to \mathbb{R}\}$

* $C^k(\Omega) = \{$f^{n}s $f: \Omega \to \mathbb{R}$ s.t. f and all its derivatives/partial derivatives of order $\leq k$ exist & are cts on $\Omega\}$.

Note: 0th order derivative of f is f.

* $C^\infty(\Omega) = \{f : \Omega \to \mathbb{R}$ s.t. f & all derivs/partials of all orders exist & are cts.$\}$, a.k.a. smooth f^{n}s.

$f \in C^\omega(\Omega)$ says, at every pt. $x_0 \in \Omega$:

(i) f has a power series expansion about x_0 (namely, its Taylor series) &

(ii) f is given by its power series expansion, i.e., the power series converges to f on some nbhd of x_0.

$C^\omega(\Omega)$ = real analytic f^{n}s on Ω.

(i) $\Rightarrow f \in C^\infty(\Omega)$;

(i) $\not\Rightarrow$ (ii) in $\mathbb{R}^n$.

Note $f(x) = |x|$ is in $C^0(\mathbb{R})$, not in $C^1(\mathbb{R})$.
Using f^ns o.t.f. $x \mapsto x^{3/2}$, $x \mapsto x^{5/2}$ etc, you can show $C^m(\Omega) \subsetneq C^{m-1}(\Omega)$.

Consider

$$f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Check: * $f^{(n)}(x)$ exists $\forall x \neq 0$.

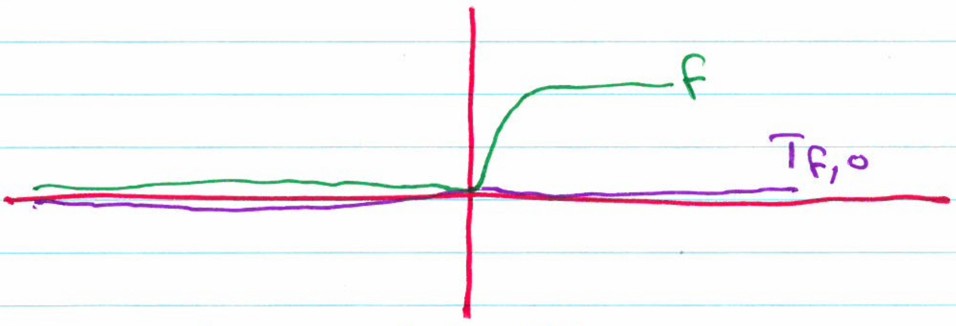
nth derivative of f

* $f^{(n)}(0) = 0 \quad \forall n$

* $f^{(n)}$ is cts on $\mathbb{R}$.

$\Rightarrow$ Taylor series for f at 0 is

$$T_{f,0}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \equiv 0.$$



f is not equal to $T_{f,0}$ on any nbhd of 0.
$f \in C^\infty(\Omega)$, $f \notin C^\omega(\Omega)$.

$$C^\omega \subsetneq C^\infty \subsetneq \cdots C^{1337} \subsetneq C^{1336} \cdots\cdots \subsetneq C^1 \subsetneq C^0.$$