LECTURE 15

Sufficient conditions for $f'(z_0)$ to exist.

Suppose

*1 $f = u+iv$ is defined in a nbhd of $\overset{x_0+iy_0}{\overset{\shortparallel}{z_0}}$;

*2 u_x, u_y, v_x, v_y are defined & cts in a nbhd of z_0

* C/R hold at (x_0, y_0).

Then $f'(z_0)$ exists.

Rmk: *1, *2 $\Rightarrow$ f is cts in a nbhd of z_0.

Formulae (cf. $f : \mathbb{R} \to \mathbb{R}$).

- * $\frac{d}{dz}(c) = 0$ c constant
- * $\frac{d}{dz}(z^n) = n z^{n-1}$ $n \in \mathbb{Z}$
- * $\frac{d}{dz}(e^z) = e^z$
- * $\frac{d}{dz} \sin z = \cos z$; $\frac{d}{dz} \cos z = -\sin z$
- * other trig, hyperbolic etc.

For f, g diff^{ble}:

- * $(f \pm g)' = f' \pm g'$;
- * $(fg)' = fg' + f'g$;
- * $(f/g)' = \frac{gf' - fg'}{g^2}$ $g \neq 0$.

Chain rule: f diff^{ble} at z_0 & g diff^{ble} at $f(z_0)$, then $g \circ f$ is diff^{ble} at z_0 &

$$(g \circ f)'(z_0) = g'(f(z_0)) \cdot f'(z_0).$$

Write $\frac{dg}{dz} = \frac{dg}{dw} \frac{dw}{dz}$ where $w = f(z)$.

E.g. $\frac{d}{dz}(74z^2 + 9)^{171}$ $= 171(74z^2+9)^{170} \cdot 2 \cdot 74 z$

$= 25308(74z^2+9)^{170} z.$

C/R in polar coords

$z = x + iy = re^{i\theta}$

$x = r\cos\theta$, $y = r\sin\theta$ $f = u + iv$ is $\mathbb{C}$-diff^{ble}.

Chain rule (in 2D) $\Rightarrow$

$$u_r = u_x \cos\theta + u_y \sin\theta \quad ①$$

$$u_\theta = -u_x r \sin\theta + u_y r \cos\theta \quad ②$$

$$v_r = v_x \cos\theta + v_y \sin\theta \quad ③$$

$$v_\theta = -v_x r \sin\theta + v_y r \cos\theta . \quad ④$$

Combine with C/R from Lec 14 ☆

$\Rightarrow$ C/R in polar coordinates:

$$\left.\begin{aligned} r u_r &= v_\theta \\ u_\theta &= -r v_r \end{aligned}\right\}$$

Useful to note: recall Ⓚ, Lec 14

If f' exists, then $\boxed{f' = u_x + i v_x}$ ⑤

& similarly

$$\boxed{f'(z) = e^{-i\theta}(u_r + i v_r)}$$

Another approach to C/R.

Formally: $(x,y) \to (z,\bar{z})$

$$z = x+iy \ , \ \bar{z} = x - iy.$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial x}$$

$$= \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}} \qquad ⑦$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial f}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial y}$$

$$= i\frac{\partial f}{\partial z} - i\frac{\partial f}{\partial \bar{z}} \qquad ⑧$$

⑦ $-i$ ⑧ $\quad \frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y} = 2\frac{\partial f}{\partial z}$

$$\Rightarrow \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

⑦ $+i$ ⑧ $\quad$ ☆ $\Rightarrow \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

$\frac{\partial}{\partial z}$ $\frac{\partial}{\partial \bar{z}}$ are called Wirtinger operators

Ex: $f(z) = z^n = (x+iy)^n \quad n \in \mathbb{Z}$.

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(x+iy)^n$$

$$= \frac{1}{2}n(x+iy)^{n-1}(1-i^2) = n(x+iy)^{n-1}$$

$$= nz^{n-1} = f'(z)$$

$$\frac{\partial f}{\partial \bar{z}} = 0 \quad ☆$$

For $f = u+iv$ complex differentiable:

$$\frac{1}{2}\frac{\partial f}{\partial x} = \frac{1}{2}(u_x+iv_x) \overset{C/R}{=} \frac{1}{2}(v_y - iu_y)$$

$$= -\frac{i}{2}(u_y+iv_y) = -\frac{i}{2}\frac{\partial f}{\partial y}.$$

$$\Rightarrow \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0.$$

$$\Rightarrow \boxed{\frac{\partial f}{\partial \bar{z}} = 0}.$$ C/R version II.

Application: When is $g(z) = |z|^2$ diff^{ble}?

Note $g(z) = z\bar{z} = x^2+y^2$.

C/R_{II} $\frac{\partial g}{\partial \bar{z}} = 0 \iff z = 0.$

So, the only point at which g is possibly diff^{ble} is 0.

Note $g = \underbrace{(x^2+y^2)}_{u} + \underbrace{0}_{v}i$, so easy to check suff cond^{n}s are satisfied.

from Lec 15

So g is diff^{ble} precisely at 0.

Sidebar: From ⑤, for diff^{ble} f,

$$\frac{df}{dz} = u_x + iv_x = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

$$= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial f}{\partial z}$$

Def^n: $f : \overset{\subset\mathbb{C}}{\Omega} \to \mathbb{C}$ is analytic at z_0 if if it is differentiable on a neighbourhood of z_0.

A f^n. is singular at z_0 if it is NOT analytic at z_0, but is analytic at some point in every nbhd of z_0.

E.g. $z \mapsto 1/z$ is analytic on $\mathbb{C}_*$, & is singular at 0.

A f^n. is entire if is is analytic on $\mathbb{C}$, e.g. e^z, polynomials, sin, cos, sinh, cosh, etc.