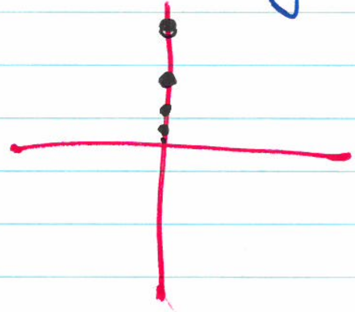
LECTURE 13

* An open, connected subset of $\mathbb{C}$ is called a domain.

* A set whose interior is a domain is called a region. (B-C terminology).

* A pt $z \in \mathbb{C}$ is called an accumulation point of a set $\Omega \subseteq \mathbb{C}$ if every deleted nbhd of z intersects Ω.

E.g. ① $\Omega = \{ i/2^n \}_{n \in \mathbb{N}_0}$



0 is the only accumulation pt. of Ω.

E.g. ② $\Omega = B_1$, $\overline{B}_1$ = set of accumulation pts.

§15-16 Limits

Let f be a $\mathbb{C}$-valued f^n, defined on a deleted nbhd of some $z_0 \in \mathbb{C}$.

$\lim_{z \to z_0} f(z) = w_0$, i.e., "the limit as z approaches z_0 of $f(z)$ is w_0" says:

Given $\varepsilon > 0$, $\exists \delta > 0$ s.t.

$$0 < |z - z_0| < \delta \Rightarrow |f(z) - w_0| < \varepsilon.$$

Note: f does not have to be defined at z_0.

E.g. $f(z) = \begin{cases} 0 & z \neq 0 \\ 1337 & z = 0 \end{cases}$:

$$\lim_{z \to 0} f(z) = 0;$$

E.g. $\lim_{z \to 0} \frac{\sin z}{z} = 1.$

<u>RMK</u>: if a limit exists, it is unique.

<u>PF</u>: in tutes, wk 5.

§17 (8 Ed §16) Limit Theorems

Suppose $f(\underset{x+iy}{z}) = u(x,y) + i v(x,y)$

Put $z_0 = x_0 + i y_0$, $w_0 = u_0 + i v_0$

Then

$$\lim_{z \to z_0} f(z) = w_0 \Leftrightarrow \begin{cases} \lim_{(x,y) \to (x_0,y_0)} u(x,y) = u_0 \\ \& \\ \lim_{(x,y) \to (x_0,y_0)} v(x,y) = v_0 \end{cases}$$

Thm: 2 Suppose $\lim_{z\to z_0} f(z) = w_0$ & $\lim_{z\to z_0} g(z) = \xi_0$, & $\lambda \in \mathbb{C}$. Then:

1. $\lim_{z\to z_0} (f \pm g)(z) = w_0 \pm \xi_0$;
2. $\lim_{z\to z_0} (\lambda f)(z) = \lambda w_0$;
3. $\lim_{z\to z_0} (fg)(z) = w_0 \xi_0$;
4. $\lim_{z\to z_0} \left(\frac{f}{g}\right)(z) = w_0/\xi_0$ as long as $\xi_0 \neq 0$.

Limits involving ∞

Recall in $\mathbb{R}$:

$\lim_{x\to x_0} f(x) = \infty$ means: Given $M > 0$ $\exists\, \delta > 0$ s.t. $0 < |x - x_0| < \delta \Rightarrow f(x) > M$.

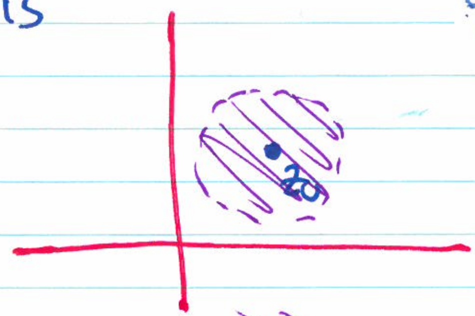
Analogously for $\lim_{x\to x_0} f(x) = -\infty$ ☆

Similarly $\lim_{x\to\infty} f(x) = \lambda \in \mathbb{R}$ says:

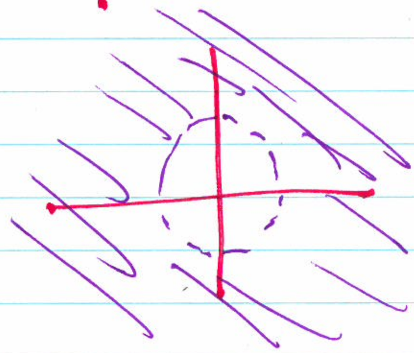
Given $\varepsilon > 0$ $\exists\, M > 0$ s.t. $x > M \Rightarrow |f(x) - \lambda| < \varepsilon$.

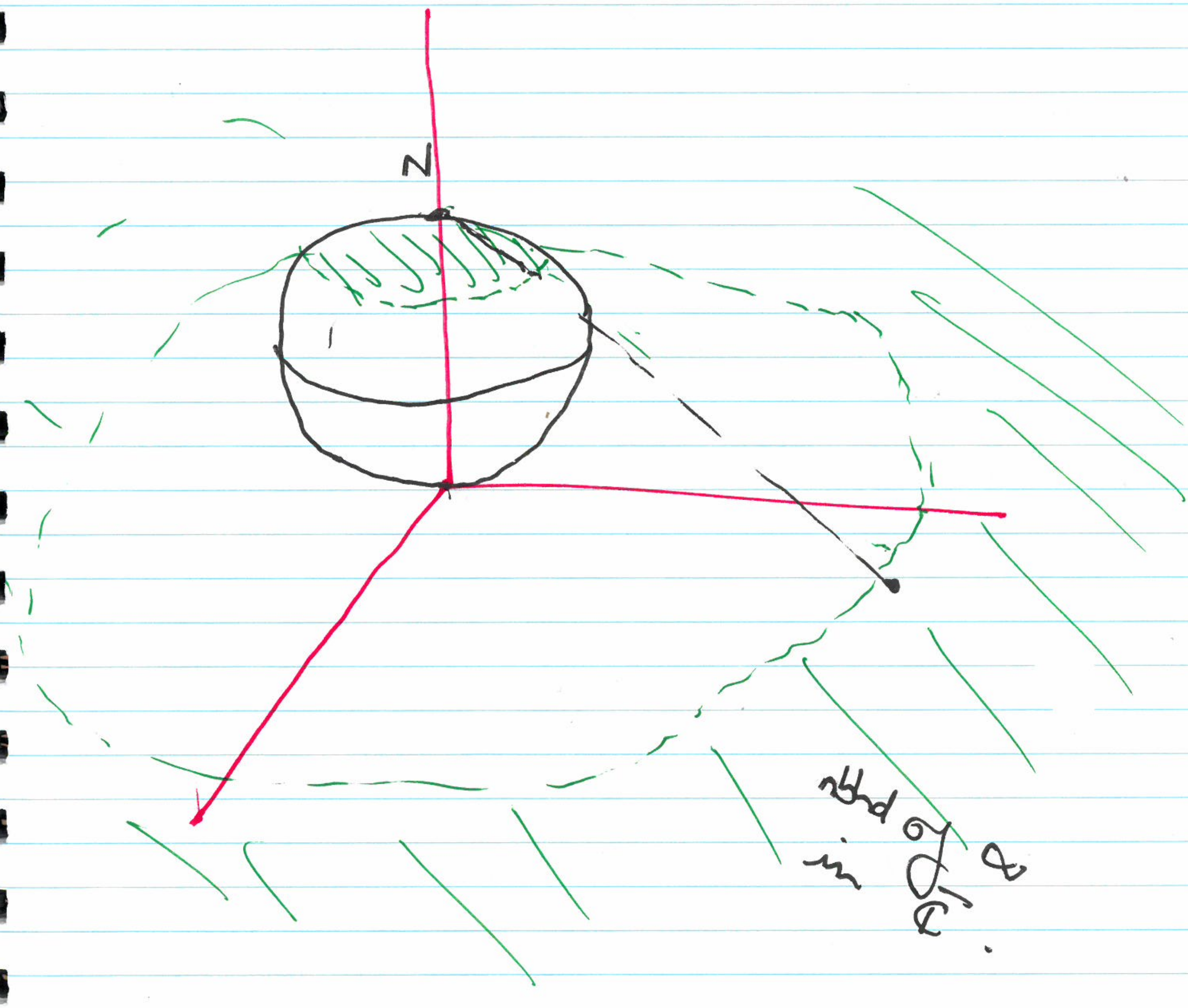
Define $\lim_{x\to-\infty} f(x) = \lambda$, $\lim_{x\to\infty} f(x) = \infty$ analogously. ☆

In $\mathbb{C}$ a nbhd of $z_0 \in \mathbb{C}$ is



A nbhd of ∞ in $\hat{\mathbb{C}}$ has the form $\{z : |z| > M\}$





"close to ∞" $\iff$ $|z|$ is large $\iff$ $1/|z|$ is small.

Keeping that in mind:

* $\lim_{z \to z_0} f(z) = \infty$ means $\lim_{z \to z_0} \frac{1}{f(z)} = 0$.

* $\lim_{z \to \infty} f(z) = w_0 \in \mathbb{C}$ means $\lim_{z \to 0} f(1/z) = w_0$.

* $\lim_{z \to \infty} f(z) = \infty$ means $\lim_{z \to 0} \frac{1}{f(1/z)} = 0$.

E.g. ① Show $\lim_{z \to -1} \frac{iz+3}{z+1} = \infty$; $= f(z)$

② Find $\lim_{z \to \infty} \frac{2z^3-1}{z^2-1}$. $= g(z)$

① f is defined on $\mathbb{C} \setminus \{-1\}$

To show the limit, WTS $\lim_{z \to -1} \frac{1}{f(z)} = 0$ (*)

$$\frac{1}{f(z)} = \frac{1}{\frac{iz+3}{z+1}} = \frac{z+1}{iz+3}$$

$$\text{LHS of } (*) = \lim_{z \to -1} \frac{1}{f(z)} = \lim_{z \to -1} \left(\frac{z+1}{iz+3} \right) = \frac{\lim_{z \to -1} (z+1)}{\lim_{z \to -1} (iz+3)} = \frac{0}{3-i} = \text{RHS of } (*)$$

② Claim: $\lim_{z\to\infty} g(z) = \infty$

So: WTS: $\lim_{z\to 0} 1/g(1/z) = 0$.

$$1/g(1/z) = 1 \Big/ \frac{2(1/z)^3 - 1}{(1/z)^2 - 1}$$

$$= \frac{(1/z)^2 - 1}{2(1/z)^3 - 1} \cdot \frac{z^3}{z^3}$$

$$= \frac{z - z^3}{2 - z^3} \to 0/2 \text{ as } z \to 0 \text{ as req}^d.$$

§19 ff (8 Ed §18 ff) Continuity

f $\mathbb{C}$-valued, defined in a nbhd of $z_0 \in \mathbb{C}$.

f is continuous at z_0 $\boxed{\lim_{z\to z_0} f(z) = f(z_0)}$, if

given $\varepsilon > 0$ $\exists \delta > 0$ s.t.

$|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$.

Basic results:

① If $f: \Omega \to U$ & $g: U \to W$ are cts, so is $g \circ f: \Omega \to W$ (g composed with f, $(g\circ f)(z) = g(f(z))$).

② If f is cts & nonzero at z_0, then $\exists \varepsilon > 0$ s.t. $f(z) \neq 0$ on $B_\varepsilon(z_0)$. Pf: tutes, wk 5.

③ $f: x+iy \mapsto u(x,y) + i v(x,y)$ is cts $\Leftrightarrow$ u & v are cts.

④ Obvious analogues of Thm's 1 & 2 from §17 hold.

Next: differentiability in $\mathbb{C}$, Cauchy-Riemann.