LECTURE 6

For a general Möb transformation

$$T(z) = w = \frac{az+b}{cz+d} \qquad ad-bc \neq 0 \qquad (*)$$

$(*)$ can be rewritten as

$$Azw + Bz + Cw + D = 0 \qquad (**)$$

(with $A = c$, $B = -a$, $C = d$ & $D = -b$) called the implicit form.

Möbius transforms are 1-1 & onto.

Case I $c = 0$: from Lecture 5, T is a bijection (1-1 & onto) $\mathbb{C} \to \mathbb{C}$.

Case II $c \neq 0$:

argue by the composition from Lecture 5, or note that $(*) \Rightarrow z = \frac{-dw+b}{cw-a}$,

i.e. $T^{-1}(w) = \frac{-dw+b}{cw-a}$

$\Rightarrow$ in case II, T is 1-1 & onto, $\mathbb{C} \setminus \{-d/c\} \to \mathbb{C} \setminus \{a/c\}$.

Q^n: can we extend T to a f^n: $\mathbb{C} \to \mathbb{C}$ in case II?

Ans: yes, e.g., take $T(-d/c) = a/c$.

(indeed taking $T(-d/c) =$ any value in $\mathbb{C}$ answers the q^n with "yes", but taking $T(-d/c) = a/c$ makes the extension 1-1 & onto.

Deeply unsatisfying, because this extension is discontinuous at $-d/c$ (intuitively clear, because $|w| \to \infty$ as $z \to -d/c$, but $|T(-d/c)| = |a/c|$, which is finite; will be made rigorous later).

IMPORTANT CONCEPT:

Extend $\mathbb{C}$ to the extended complex plane, denoted by $\overline{\mathbb{C}}$ by "adding a point at infinity", which we call ∞.

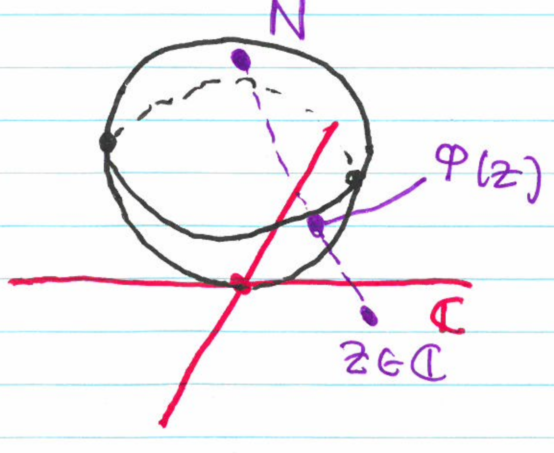
Define $T(-d/c) = \infty$ & $T(\infty) = a/c$.

This extends T to a map $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$, which is 1-1 & onto.

RMK 1 $\overline{\mathbb{C}}$ is a topological space, & the given extension is continuous.

RMK 2 In Case I ($c=0$), extend T to a map $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ by setting $T(\infty) = \infty$.

To visualise (and calculate) in $\overline{\mathbb{C}}$:



φ maps $\mathbb{C}$ onto the (surface of the) given sphere, called the Riemann sphere.