LECTURE 5

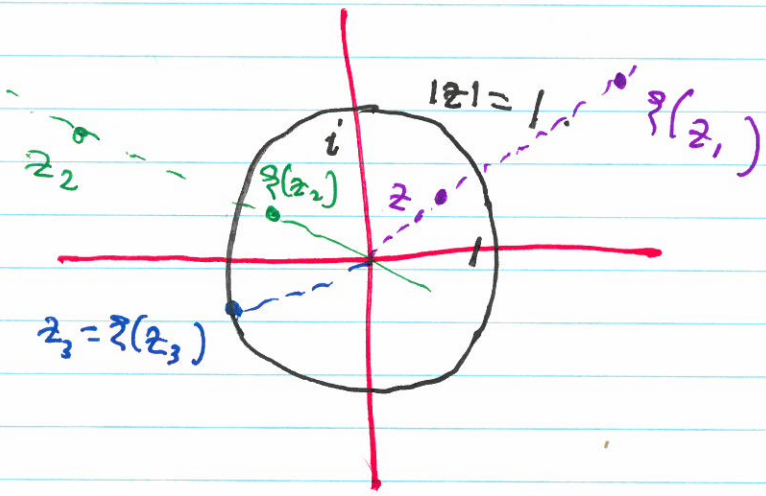
Ex2: $z \mapsto 1/z$ on $\mathbb{C}_*$

Define $\xi(z) = z/|z|^2$ on $\mathbb{C}_*$;

composed with $\eta(\xi) = \bar{\xi}$ on $\mathbb{C}$.

$$\eta \circ \xi(z) = \eta(\xi(z)) = \overline{\xi(z)} = \overline{\left(z/|z|^2\right)}$$

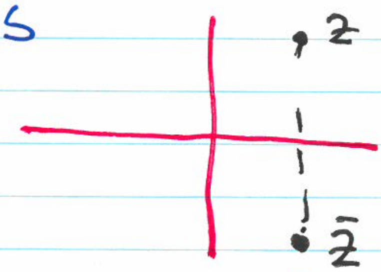
$$= \bar{z}/|z|^2 = \cancel{\bar{z}}/(\cancel{\bar{z}} z) = 1/z. \quad z \in \mathbb{C}_*$$



with respect to

ξ is called inversion (w.r.t. the unit circle)

η is reflection in the $\mathbb{R}$ axis



So $z \mapsto \frac{1}{z}$ is inversion, then reflection in the $\mathbb{R}$ axis.

For $w = 1/z = \bar{z}/|z|^2$

$x+iy \mapsto u+iv$

$w = \dfrac{x-iy}{x^2+y^2}$, so

$u = \dfrac{x}{x^2+y^2}$ (1) & $v = \dfrac{-y}{x^2+y^2}$ (2)

We can use this to show:

$\boxed{\text{$1/z$ maps circles & lines in the z-plane to circles & lines in the w-plane.}}$

Note: circles & lines in the z plane have the form

$$\left.\begin{aligned} &A(x^2+y^2)+Bx+Cy+D=0 \\ &A,B,C,D \in \mathbb{R}, \quad B^2+C^2 > 4AD \end{aligned}\right\} \quad (*)$$

$A=0$ line, $A \neq 0$ circle.

From (*), we have

$$\left(x+\frac{B}{2A}\right)^2 + \left(y+\frac{C}{2A}\right)^2 = \left(\frac{\sqrt{B^2+C^2-4AD}}{2A}\right)^2$$

$(A \neq 0)$

For $w = 1/z$, (1) & (2) $\Rightarrow$

$D(u^2+v^2)+Bu-Cv+A=0.$ ✡

Terminology reminder:

* $f: \Omega \to \mathbb{C}$ is 1-1 or injective says:
  $f(z) = f(\xi) \Rightarrow z = \xi$.

* $f: \Omega \to \Lambda \subseteq \mathbb{C}$ is onto or surjective says:
  given $\eta \in \Lambda$, $\exists$ (at least one) $z \in \Omega : f(z) = \eta$.
  there exists.

Möbius transformations: B-C §99 (8Ed §93ff.)

Def^{n}: Let $a,b,c,d \in \mathbb{C}$ with $\boxed{ad-bc \neq 0}$ ③

Then $\boxed{w = T(z) = \dfrac{az+b}{cz+d}}$ ④

is a Möbius (a.k.a. linear fractional) transformation

Natural domain of definition:

* $c=0$: $\mathrm{dom}(w) = \mathbb{C}$ (note $c=0 \overset{③}{\Rightarrow} d \neq 0$)
* $c \neq 0$ $\mathrm{dom}(w) = \mathbb{C} \setminus \{-d/c\}$.

Goal: understand T geometrically.

CASE I $c=0$.

Claim: T map $\mathbb{C} \to \mathbb{C}$ 1-1 & onto.

Pf: (i) 1-1 suppose $T(z)=T(\xi)$ (**)

(WTS): $z=\xi$. (**) $\Rightarrow$ $\frac{a}{d}z+\frac{b}{d} = \frac{a}{d}\xi+\frac{b}{d}$.

want to show

$-b/d$, $\times d/a$ $\Rightarrow z=\xi$ ∎

(ii) onto: given $w \in \mathbb{C}$, need $z \in \mathbb{C}$ s.t. $T(z)=w$.

Check: $z = \frac{d}{a}(w - b/d)$ works ★ ∎

Rmk: $z \mapsto \underbrace{\frac{a}{d}}_{A} z - \underbrace{b/d}_{\text{"}b\text{"}}$

is a linear transformation as previously covered.

CASE II $c \neq 0$

$$\underbrace{w}_{T(z)} = \frac{az+b}{cz+d} = \frac{a(z+d/c) - ad/c + b}{c(z+d/c)}$$

$$= a/c + \frac{bc-ad}{c} \cdot \frac{1}{cz+d}.$$

So in case II, T is the composition of:

$Z_1 = cz+d$;

$W = 1/Z_1$;

$w = a/c + \frac{bc-ad}{c} W$.

So in both cases I & II, T is a composition of maps previously studied.