LECTURE 2

Further motivation:

* (i) $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots = \pi^2/12$ ✓

(iii) $\sum_{k=1}^{\infty}\frac{1}{1+4k^2\pi^2} = \frac{1}{2}\left(\frac{1}{e-1}-\frac{1}{2}\right).$

Re (i) $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$: Riemann zeta f^n. $s\in\mathbb{C}$.

\zeta

Riemann hypothesis: ζ has ∞ly many non-trivial zeros, & they all lie on the line $\mathrm{Re}(s)=\frac{1}{2}$.

Note that def^n as given only makes sense for $\mathrm{Re}(s)>1$: need to extend ζ to $\mathbb{C}$, done by analytic continuation

* trivial zeros $-2,-4,-6,\ldots$
* Millenium Problems

Complex Numbers:

* 1545 Cardano (Ars Magna)
  real roots of $x^3 + ax + b$
  $5 + \sqrt{-15}$ "mental torture"
* 1575 i : rules for operations in $\mathbb{C}$
* 1629 $a + \sqrt{-b}$ "solutions impossibles"
* Descartes : imaginary numbers
* 1831 complex #s.

INTRO BC §1–3

* $\mathbb{N}$: $\{1, 2, 3 \ldots\}$ natural #s
* $\mathbb{N}_0$: $\{0, 1, 2, \cdots\}$
* $\mathbb{Z}$: $\{0, \pm 1, \pm 2, \ldots\}$ integers
* $\mathbb{Q}$: $\{p/q : p, q \in \mathbb{Z}, q \neq 0\}$.
* $\mathbb{R}$: real #s
* $\mathbb{C}$: Complex #s

Various equivalent representations of $\mathbb{C}$:

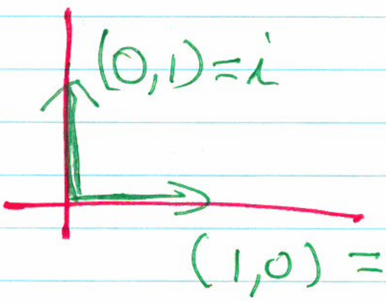
$$z = (\underset{\mathbb{R}}{x}, \underset{\mathbb{R}}{y}) \in \text{complex plane.}$$

So $z = x(1,0) + y(0,1)$.

$= x + iy$

real part of z

imaginary part of z



"i is the complex number represented by $(0,1)$"

We say $\mathbb{R} \subset \mathbb{C}$ by identifying the complex # $\boxed{x + 0i}$ with the real # $\boxed{x}$.

Addition in $\mathbb{C}$: $+$

Set $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$, i.e.,

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

Multiplication in $\mathbb{C}$:

$\times$ or $\cdot$ or juxtaposition:

$$(x_1, y_1)\cdot(x_2, y_2) = (x_1x_2 - y_1y_2,\ y_1x_2 + x_1y_2)$$

$$(x_1 + iy_1)\cdot(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2).$$

Note def^n of $\times$ formally applies if we use the usual rules for algebra in $\mathbb{R}$, & set $i^2 = -1$.

$\boxed{\text{With this } + \text{ \& } \times,\ \mathbb{C} \text{ is a field}}$

Check: $\mathbb{C}$ is closed under $+$ & $\times$.

F2) (i) $0 = 0 + 0i$;

(ii) $-z = -(x+iy) = (-x) + i(-y)$.

F5) (i) $1 = 1 + 0i$;

(ii) $z = x + iy \neq 0$;

$$z^{-1} = \frac{1}{x+iy}\cdot\frac{x-iy}{x-iy} = \frac{x}{x^2+y^2} - i\,\frac{y}{x^2+y^2}$$

Since $\mathbb{C}$ is a field, there holds:

$$z_1 z_2 = 0 \Rightarrow z_1 = 0 \text{ or } z_2 = 0 \text{ or both,}$$

called null-factor or cancellation law