Claim \( \frac{1}{2} \) maps circles &
lines in the \( z \)-plane to
circles & lines in the \( w \)-plane.

Note: circles & lines in the
\( z \)-plane can be represented as
\[ A(x^2+y^2) + Bx + Cy + D = 0 \]

\[ \text{Check (tutes) for } f(z) = \frac{1}{2}, \]
\[ D(u^2+v^2) + Bu - Cv + A = 0. \]
Terminology

\[ f : \mathbb{R} \rightarrow \mathbb{C} \] is \textbf{H-1} or \textbf{injective} if \[ f(x) = f(y) \Rightarrow x = y. \]

\[ f : \mathbb{R} \rightarrow \mathbb{C} \] is \textbf{onto} or \textbf{surjective} if \[ f(x) \in \mathbb{C} \] for all \[ x \in \mathbb{R}. \]
Given \( n \in \mathbb{E} \) (at least one), \( z \in \mathbb{J} : f(z) = n \).

Moebius transformations.

Definition: Let \( a, b, c, d \in \mathbb{C} \) with \( ad - bc \neq 0 \).

Then

\[
W = T(z) = \frac{az + b}{cz + d} \tag{2}
\]

is a Moebius (a.k.a. linear fractional) transformation.

Natural domain of def:

- \( c = 0 : \text{dom}(f) = \mathbb{C} \setminus \{ -\frac{d}{a} \} \)
- \( c \neq 0 : \text{dom}(f) = \mathbb{C} \setminus \emptyset \)
Note: (2) can be rewritten as \(Aw + Bz + Cw + D = 0\) (with \(A = c, B = -a, C = d, D = -b\)), (Implicit form).

Goal: understand \(T\)

Case I: \(c = 0\) \((\Rightarrow ad \neq 0 \oplus)\)

Claim: \(T\) maps \(c \rightarrow c\)

1-1 & onto.

\[ Pf: 1-1, \text{ Suppose} \]
\[ T(x) = T(y) \]
\[ \text{Want to show,} \]
\[ \text{WTS:} \quad z = \frac{b}{a} \]
\[ \Rightarrow z = \delta. \]
(iv) Given \( w \in C \), need \( z \in C \) s.t. 
\( T(z) = w \).

Check: \( z = \frac{d}{a}(w - \frac{1}{a}) \)

Works.

CASE II \( c \neq 0 \).

\[
\begin{align*}
    w &= \frac{az + b}{c^2 + d} = \frac{a(z + \frac{a}{c}) - \frac{ad}{c} + b}{c(z + \frac{a}{c})} \\
    &= \frac{\frac{a}{c} + \frac{bc - ad}{c}}{c^2 + d} \\
    &= \frac{a}{c} + \frac{bc - ad}{c} \\
\end{align*}
\]

So, in both cases I and II, \( T \) is the composition of maps previously studied.
\[ W = \sqrt{2}; \quad w = \frac{a}{c} + \frac{bc-ad}{c} \]

So, in both cases 1 & 2 \( T \) is a composition of maps previously studied.

Note in case 1:

\[ \Rightarrow z = \frac{-dw+b}{cw-a} \]

i.e. \( T^{-1}(w) = \frac{-dw+b}{cw-a} \)

\( \Rightarrow T \) is 1-1 & onto.

\( \Rightarrow \) \( C \rightarrow C \) & \( \mathbb{C} \rightarrow \mathbb{C} \)
Can we extend $T$ to a fn. $C \to C$ in case II?
Yes: set $T(-\infty) = \infty$, but note this is a discontinuous extension.

IMPORTANT: extend $C$ to the extended complex plane, denoted $\mathbb{C} \cup \{\infty\}$.

"Adding a point at infinity," which we call $\infty$.

Define $T(-\infty) = \infty$ \& $T(\infty) = 9 \infty$ in case II.

2. In case I set $T(\infty) = \infty$.

This extends $T$ to a map $C \to C$.
Friday

C.T.5 extension

Riemann sphere