MATH1051

CALCULUS
AND
LINEAR ALGEBRA I

Semester 1, 2008
Lecture Workbook Solutions

How to use this workbook

This book should be taken to lectures, tutorials and lab sessions. In the lectures, you will be expected to fill in the blanks in any incomplete definitions, theorems etc, and any examples which appear. The lecturer will make it clear when they are covering something that you need to complete.

The completed workbook will act as a study guide designed to assist you in working through assignments and preparing for the mid-semester and final exams. For this reason, it is very important to attend lectures.

The text for the calculus part of the course is "Calculus" by James Stewart, 6th edition, 2008. We often refer to the text in this workbook, and many of the definitions, theorems, and examples come from the text. (References are also provided for the 5th edition). It is strongly advised that you purchase a copy.

Although there is no set text for the linear algebra part of the course, there are several books in the library which cover the material well (see the course website).

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1   Complex Numbers

1.1   Notation

The symbol $\in$ means "is an element of". Thus we write

$$x \in \mathbb{R}$$

to mean x is an element of $\mathbb{R}$. That is, x is a real number. Hence, we can write, for example, $2 \in \mathbb{R}$, $\pi \in \mathbb{R}$, $-\sqrt{3} \in \mathbb{R}$, and so forth.

1.2   Complex Numbers

Complex numbers were introduced in the 16th century to obtain roots of polynomial equations. A complex number is of the form

$$z = x + iy$$

where $x, y \in \mathbb{R}$ and i is (formally) a symbol satisfying $i^2 = -1$. The quantity x is called the *real part* of z and y is called the *imaginary part* of z.

The set of all complex numbers is denoted $\mathbb{C}$. Thus $3 - 2i \in \mathbb{C}$.

1.2.1   Example

The real part of $3 - 2i$ is 3 and the imaginary part is -2 (not $-2i$).

Complex numbers can be added and multiplied by replacing i^2 everywhere with -1. For example $(2i)^2 = 4i^2 = -4$.

1.2.2   Example

Simplify $(3 - 2i)(1 + i)$.

$$
\begin{aligned}
(3 - 2i)(1 + i) &= 3 + 3i - 2i - 2i^2 \\
&= 3 + i + 2 = 5 + i
\end{aligned}
$$

1.2.3   Example

Suppose $a, b \in \mathbb{R}$. Simplify $(a + bi)(a - bi)$.

$$
\begin{aligned}
(a + bi)(a - bi) &= a^2 - abi + abi - b^2 i^2 \\
&= a^2 + b^2
\end{aligned}
$$

1.2.4 Example

Simplify $\dfrac{3 - 2i}{1 - i}$.

$$
\begin{aligned}
\frac{3 - 2i}{1 - i} &= \frac{3 - 2i}{1 - i} \times \frac{(1 + i)}{(1 + i)} \\
&= \frac{(3 - 2i)(1 + i)}{1^2 + 1^2} \\
&= \frac{1}{2}(3 - 2i)(1 + i) \\
&= \frac{1}{2}(5 + i) \\
&= \frac{5}{2} + \frac{1}{2}i
\end{aligned}
$$

It is a fact that if we consider complex roots of polynomials and count them with their correct multiplicity, then a polynomial of degree n always has n roots. For example, every quadratic has two roots.

1.2.5 Example

Find the roots of $x^2 + 2x + 2 = 0$.

$$
\begin{aligned}
x^2 + 2x + 2 &= (x + 1)^2 + 1 = 0 \\
\iff (x + 1)^2 &= -1 = i^2 \\
\iff x + 1 &= \pm i \\
\therefore x &= -1 \pm i \text{ are the roots.}
\end{aligned}
$$

Alternatively use the quadratic formula:

$$
\begin{aligned}
x &= \frac{1}{2}(-2 \pm \sqrt{4 - 8}) = \frac{1}{2}(-2 \pm 2i) \\
&= -1 \pm i.
\end{aligned}
$$

1.3 Polar form

A complex number $z = x + iy$ may be represented by a point in the *complex plane* where the vertical axis is the *imaginary axis* and the horizontal axis is the *real axis*.

We can also specify z by giving the length r and the angle θ in figure 1. The quantity r is called the *modulus* of z, denoted $|z|$. It measures the distance of z from the origin. The angle θ is called the *argument* of z. We have:
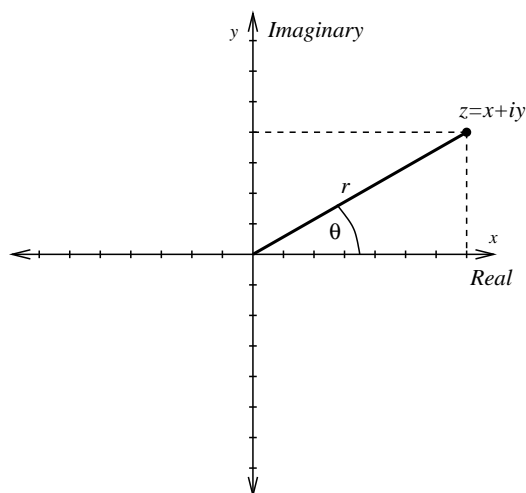
Figure 1: A complex number can be represented in rectangular or polar form.

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
\Rightarrow z = x + iy &= r(\cos\theta + i\sin\theta).
\end{aligned}
$$

$$
\begin{aligned}
\text{Also} \quad r = |z| &= \sqrt{x^2 + y^2} \\
\tan\theta &= \frac{y}{x} \quad \text{if } x \neq 0.
\end{aligned}
$$

1.3.1 Example

Write $z = 1 + i$ in polar form.

First find the modulus:
$$|z| = \sqrt{1 + 1} = \sqrt{2}.$$

We have for the argument $\tan\theta = \dfrac{y}{x}$, so we can take

$$\begin{aligned}
\theta &= \arctan\frac{y}{x} \\
&= \arctan 1 \\
&= \frac{\pi}{4}
\end{aligned}$$

$$\therefore \quad z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right).$$

1.4 Euler's formula

Euler's formula states for any real number θ:

$$\cos\theta + i\sin\theta = e^{i\theta}.$$

(To make sense of this, one has to define the exponential function for complex arguments. This may be done using a series.)

Thus every complex number $z = x + iy$ can be represented in polar form

$$z = re^{i\theta}.$$

2 Vectors

- A *vector* quantity has both a magnitude and a direction; force and velocity are two examples of vector quantities.

- A *scalar* quantity has only a magnitude (it has no direction); time, area and temperature are examples of scalar quantities.

2.1 Revision

A vector is represented geometrically in the (x, y) plane or in (x, y, z) space by a directed line segment (arrow). The direction of the arrow is the direction of the vector, and the length of the arrow is proportional to the magnitude of the vector. Only the length and direction of the arrow are significant: it can be placed anywhere convenient in the (x, y) plane.
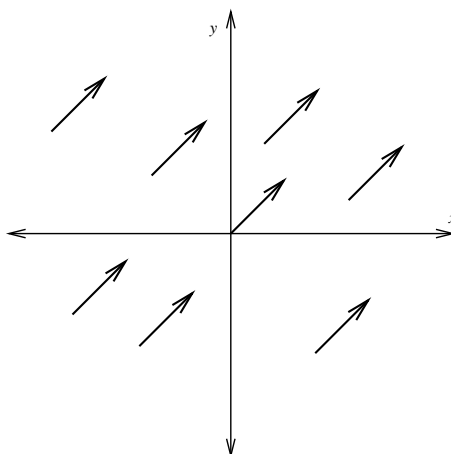
Figure 2: Vector of unit length at $45°$ to x-axis has many representations as shown.

If P, Q are points in 2-space or 3-space, $\overrightarrow{PQ}$ denotes the vector from P to Q.

A vector $\boldsymbol{v} = \overrightarrow{PQ}$ in the (x, y) plane may be represented by a pair of numbers

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} x_Q - x_P \\ y_Q - y_P \end{pmatrix}$$

which is the same for all representations $\overrightarrow{PQ}$ of $\boldsymbol{v}$. We call v_1, v_2 the *components* of the vector $\boldsymbol{v}$.

We call the vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ the *zero vector*. It is denoted by $\boldsymbol{0}$.
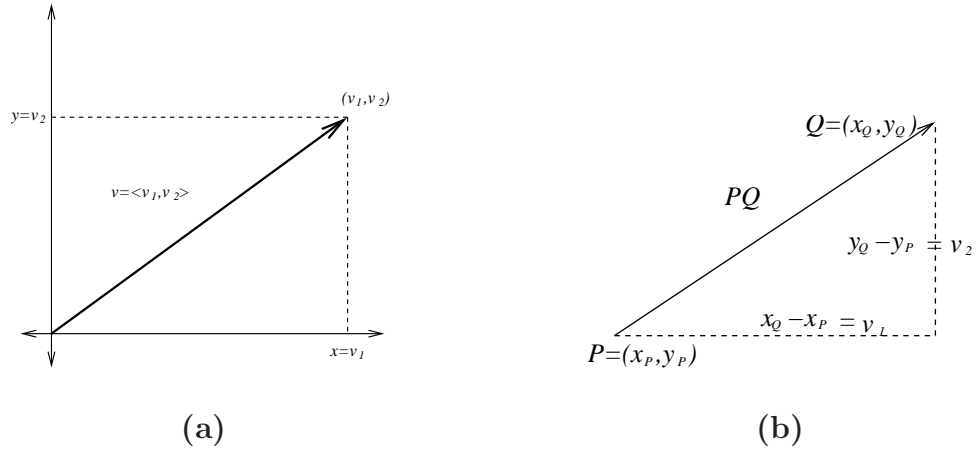
Figure 3: Geometric representation of a vector.

2.1.1 Position Vectors

Let $P = (x_p, y_p)$ be a point in the (x, y) plane. Then the vector $\overrightarrow{OP}$, where O is the origin, is called the *position vector* of P. Obviously

$$\overrightarrow{OP} = \begin{pmatrix} x_p \\ y_p \end{pmatrix}.$$

2.1.2 Definition: Magnitude

For vector $\boldsymbol{v} = \overrightarrow{PQ}$, the *magnitude* (or *length* or *norm*) of $\boldsymbol{v}$, written $\|\boldsymbol{v}\|$, is the distance between P and Q. Thus for $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ we have

$$\|\boldsymbol{v}\| = \sqrt{v_1^2 + v_2^2}.$$

2.1.3 Vector Addition

We add vectors by the triangle rule.

Consider the triangle PQR with $\boldsymbol{v} = \overrightarrow{PQ}$, $\boldsymbol{w} = \overrightarrow{QR}$. Then $\boldsymbol{v} + \boldsymbol{w} = \overrightarrow{PR}$; see Figure 4. In terms of components, if

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad \boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

then

$$\boldsymbol{v} + \boldsymbol{w} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}.$$

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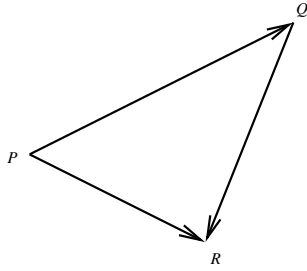
Figure 4: $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$

It follows from the component description that vector addition satisfies the following properties:

$$\begin{aligned} \boldsymbol{v} + \boldsymbol{w} &= \boldsymbol{w} + \boldsymbol{v} && \text{(commutative law)} \\ \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) &= (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} && \text{(associative law)} \\ \boldsymbol{v} + \boldsymbol{0} &= \boldsymbol{0} + \boldsymbol{v} = \boldsymbol{v} \end{aligned}$$

2.1.4 Scalar multiplication
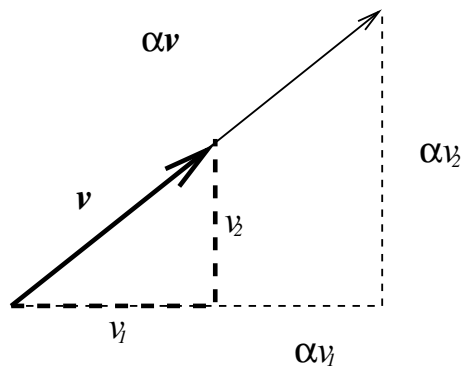
Figure 5: We can multiply the vector $\boldsymbol{v}$ by a number α (scalar).

With $\alpha \in \mathbb{R}$ a number (called a *scalar*), we define $\alpha\boldsymbol{v}$ to be the vector of magnitude

$$\|\alpha\boldsymbol{v}\| = |\alpha| \cdot \|\boldsymbol{v}\|$$

in the same direction as $\boldsymbol{v}$ if $\alpha > 0$ and opposite direction if $\alpha < 0$.

Using similar triangles it follows that if $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ then $\alpha\boldsymbol{v} = \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \end{pmatrix}$.

Note that if we multiply any vector $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ by zero we obtain the zero vector:

$$0 \cdot \boldsymbol{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \boldsymbol{0}$$

2.1.5 Unit Vectors

A *unit vector* is a vector of unit length. If $\boldsymbol{v} \neq \boldsymbol{0}$ is a vector, then

$$\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$$

determines a unit vector in the direction of $\boldsymbol{v}$.

In particular

$$\boldsymbol{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \boldsymbol{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

determine unit vectors along the x and y axes respectively.
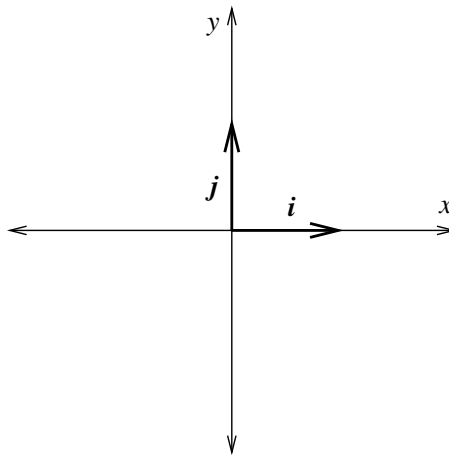
Figure 6: Unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ in the x and y directions respectively.

For any vector $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ we have

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ v_2 \end{pmatrix} = v_1 \boldsymbol{i} + v_2 \boldsymbol{j},$$

Hence we can decompose $\boldsymbol{v}$ into a vector $v_1 \boldsymbol{i}$ along the x-axis and $v_2 \boldsymbol{j}$ along the y-axis. Then v_1 and v_2 are called the *components* of $\boldsymbol{v}$ with respect to $\boldsymbol{i}$ and $\boldsymbol{j}$.

2.1.6   Vectors in 3-space

Similarly in 3-space a vector $\boldsymbol{v} = \overrightarrow{PQ}$ is represented in component form by

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} x_Q - x_P \\ y_Q - y_P \\ z_Q - z_P \end{pmatrix}$$

which is the same for all representations $\overrightarrow{PQ}$ of $\boldsymbol{v}$.
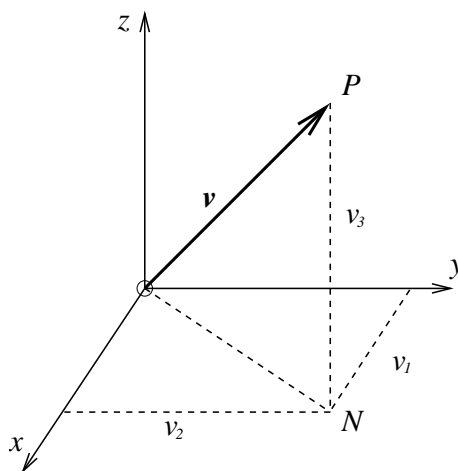
Figure 7: A vector in 3-space.

For $\boldsymbol{v} = \overrightarrow{OP} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ the *magnitude* of the vector $\boldsymbol{v}$ is

$$\|\boldsymbol{v}\| \;\; = \;\; \sqrt{ON^2 + NP^2} = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

As before, we add vectors component by component. We also define multiplication by a scalar α. So if

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \qquad \boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

are vectors then

$$\boldsymbol{v} + \boldsymbol{w} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix} \quad \text{and} \quad \alpha\boldsymbol{v} = \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \\ \alpha v_3 \end{pmatrix}, \; \text{for } \alpha \in \mathbb{R}.$$

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The unit vectors along the x, y, z axes are respectively

$$\boldsymbol{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \boldsymbol{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \boldsymbol{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Any vector $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ may be expressed as

$$\boldsymbol{v} = v_1 \boldsymbol{i} + v_2 \boldsymbol{j} + v_3 \boldsymbol{k}.$$

We call v_1, v_2, v_3 the *components* of $\boldsymbol{v}$ in the $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ directions respectively.

We usually denote 2- and 3-space by $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Thus

$$\mathbb{R}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x, y \in \mathbb{R} \right\} \quad \text{and} \quad \mathbb{R}^3 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

2.1.7 Row and column vectors

Note that we may write vectors using columns eg. $\begin{pmatrix} a \\ b \end{pmatrix}$, or as row vectors $\begin{pmatrix} a & b \end{pmatrix}$.

2.1.8 Dot Product

For non zero vectors $\boldsymbol{v} = \overrightarrow{OP}$, $\boldsymbol{w} = \overrightarrow{OQ}$ the *angle between* $\boldsymbol{v}$ and $\boldsymbol{w}$ is the angle θ with $0 \le \theta \le \pi$ radians between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ at the origin O; see Figure 8.

The *dot* (or *scalar* or *inner*) product of vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, denoted by $\boldsymbol{v} \cdot \boldsymbol{w}$, is the *number* given by

$$\boldsymbol{v} \cdot \boldsymbol{w} = \begin{cases} 0, & \text{if } \boldsymbol{v} \text{ or } \boldsymbol{w} = \boldsymbol{0} \\[2ex] \|\boldsymbol{v}\| \cdot \|\boldsymbol{w}\| \cos\theta, & \text{otherwise} \end{cases}$$

where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

If $\boldsymbol{v}$, $\boldsymbol{w} \ne \boldsymbol{0}$ and $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ then $\boldsymbol{v}$ and $\boldsymbol{w}$ are said to be *orthogonal* or *perpendicular*.

If $\boldsymbol{v} = v_1 \boldsymbol{i} + v_2 \boldsymbol{j} + v_3 \boldsymbol{k}$ and $\boldsymbol{w} = w_1 \boldsymbol{i} + w_2 \boldsymbol{j} + w_3 \boldsymbol{k}$ are two vectors, then $\boldsymbol{v} \cdot \boldsymbol{w}$ is given by:

$$\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + v_3 w_3,$$

In particular, for $\boldsymbol{v} \in \mathbb{R}^3$,

$$\|\boldsymbol{v}\|^2 = \boldsymbol{v} \cdot \boldsymbol{v} = v_1^2 + v_2^2 + v_3^2.$$
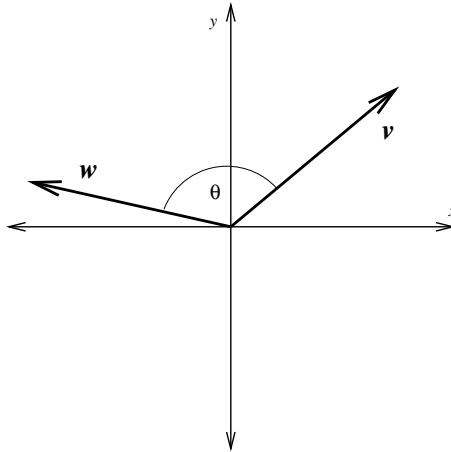
Figure 8: θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

2.1.9   The Projection Formula

Fix a vector $\boldsymbol{v}$. Given another vector $\boldsymbol{w}$ we can write it as

$$\boldsymbol{w} = \boldsymbol{w}_1 + \boldsymbol{w}_2,$$

where $\boldsymbol{w}_1$ is in the direction of $\boldsymbol{v}$ and $\boldsymbol{w}_2$ is perpendicular to $\boldsymbol{v}$; see Figure 9.
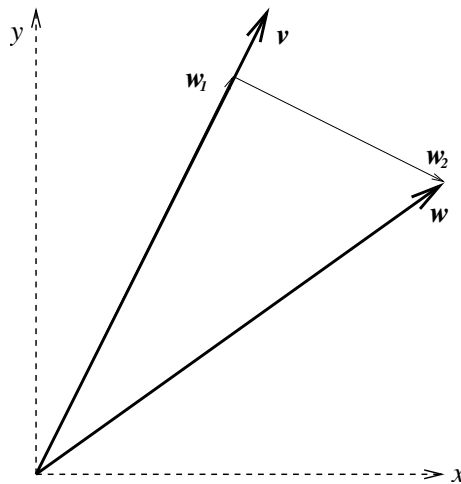
Figure 9: $\boldsymbol{w}$ can be decomposed into a component $\boldsymbol{w}_1$ in the direction of $\boldsymbol{v}$ and a component $\boldsymbol{w}_2$ perpendicular to $\boldsymbol{v}$.

Then we have the *projection formula:*

$$\boldsymbol{w}_1 = \frac{(\boldsymbol{w} \cdot \boldsymbol{v})}{\|\boldsymbol{v}\|^2}\boldsymbol{v} \quad , \quad \boldsymbol{w}_2 = \boldsymbol{w} - \frac{(\boldsymbol{w} \cdot \boldsymbol{v})}{\|\boldsymbol{v}\|^2}\boldsymbol{v}. \tag{2.1}$$

2.2 Review Problems

2.2.1 Example

A zeppelin is directed NE at 20 km/h into a 10 km/h wind in direction E60°S. Find the actual speed and direction of the airship.
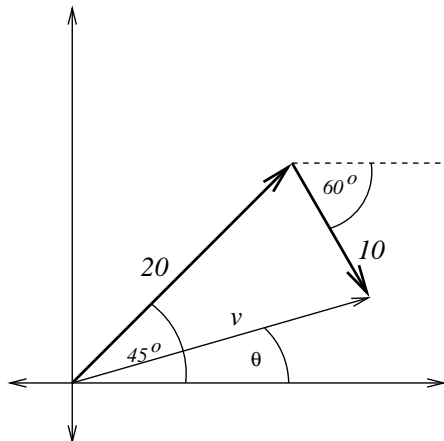
Figure 10: $\boldsymbol{v}$ gives the velocity vector of the zeppelin.

There holds:

$$
\begin{aligned}
\boldsymbol{z} &= 20\cos 45°\boldsymbol{i} + 20\sin 45°\boldsymbol{j} \\[2mm]
&= 10\sqrt{2}\boldsymbol{i} + 10\sqrt{2}\boldsymbol{j}; \\[2mm]
\boldsymbol{w} &= 10\cos 60°\boldsymbol{i} - 10\sin 60°\boldsymbol{j} \\[2mm]
&= 5\boldsymbol{i} - 5\sqrt{3}\boldsymbol{j}.
\end{aligned}
$$

So $\boldsymbol{v} = \boldsymbol{z} + \boldsymbol{w} = (10\sqrt{2} + 5)\boldsymbol{i} + (10\sqrt{2} - 5\sqrt{3})\boldsymbol{j}$

Therefore the magnitude of $\boldsymbol{v}$ is

$$\|\boldsymbol{v}\| = \sqrt{(10\sqrt{2} + 5)^2 + (10\sqrt{2} - 5\sqrt{3})^2}$$

$$= \sqrt{500 + 100\sqrt{2}(1 - \sqrt{3})}$$

$$\approx 19.91.$$

To calculate the angle θ we use

$$\tan\theta = \frac{10\sqrt{2} - 5\sqrt{3}}{10\sqrt{2} + 5}$$

$$\Rightarrow \theta \approx 15.98°$$

So $\boldsymbol{v} = 19.9\text{kmh}^{-1}$ at E16°N.

2.2.2 Example

Find the angle θ between the following pairs of vectors:

(i) $\quad \boldsymbol{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{w} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ $\qquad$ (ii) $\quad \boldsymbol{v} = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}, \quad \boldsymbol{w} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$

Note $\cos\theta = \dfrac{\boldsymbol{v} \cdot \boldsymbol{w}}{\|\boldsymbol{v}\| \cdot \|\boldsymbol{w}\|}$. So:

(i)

$$
\begin{aligned}
\|\boldsymbol{v}\| &= 1 \\
\|\boldsymbol{w}\| &= \sqrt{0 + 4 + 4} = 2\sqrt{2} \\
\boldsymbol{v} \cdot \boldsymbol{w} &= 0 + 0 + 2 = 2 \\
\therefore \quad \cos\theta &= \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} \\
\Rightarrow \theta &= \frac{\pi}{4}
\end{aligned}
$$

(ii) $\boldsymbol{v} \cdot \boldsymbol{w} = 1 \cdot 3 - 5 \cdot 3 + 4 \cdot 3 = 0$

So the vectors are perpendicular, i.e. $\theta = \dfrac{\pi}{2}$.

2.2.3 Example

If $P = (2, 4, -1)$, $Q = (1, 1, 1)$, $R = (-2, 2, 3)$, find the angle $\theta = PQR$.

$$
\begin{aligned}
\overrightarrow{QP} &= \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \\
\overrightarrow{QR} &= \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}.
\end{aligned}
$$

$\therefore \quad \left\| \overrightarrow{QP} \right\| = \left\| \overrightarrow{QR} \right\| = \sqrt{1 + 9 + 4} = \sqrt{14}.$

$$\overrightarrow{QP} \cdot \overrightarrow{QR} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = -3 + 3 - 4 = -4$$

$$\therefore \quad \cos\theta = \frac{\overrightarrow{QP} \cdot \overrightarrow{QR}}{\left\|\overrightarrow{QP}\right\| \cdot \left\|\overrightarrow{QR}\right\|} = \frac{-4}{14} \Rightarrow \theta \simeq 107°.$$

3 Matrices and Linear Transformations

3.1 2×2 matrices

Recall that a 2×2 array of numbers

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is called a 2×2 *matrix.* Suppose $\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ is a 2 element column vector. Then the *product* $A\boldsymbol{v}$ is defined to be the 2 element column vector whose entries are given by taking the dot product of each row of A with the vector $\boldsymbol{v}$:

$$A\boldsymbol{v} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

3.1.1 Example

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, and let $\boldsymbol{v} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$. Calculate $A\boldsymbol{v}$.

$$
\begin{aligned}
A\boldsymbol{v} &= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} \\
&= \begin{pmatrix} 1 \cdot 5 + 2 \cdot 6 \\ 3 \cdot 5 + 4 \cdot 6 \end{pmatrix} \\
&= \begin{pmatrix} 17 \\ 39 \end{pmatrix}.
\end{aligned}
$$

3.2 Transformations of the plane

- In computer graphics there is often a need to transform a set of points, or a figure, that appears on the screen. For example, moving the figure to another part of the screen, rotating the figure about a central point, reflecting the figure in a line, scaling the figure to a new size, or shooting at moving things.

- Moving a figure about in the (x, y)-plane can be described by the addition of a vector (the translation vector) to each of the position vectors of the points making up the figure.

- Transformations such as rotations, reflections and scalings, are examples of a special class of transformations called *linear transformations.*

- A linear transformation of the (x, y)-plane can be represented by a 2×2 matrix, and we can use matrix multiplication to calculate the effect of the transformation on the points of the plane (using position vectors).

- Transformations of the plane have important applications in computer graphics and other areas.

3.2.1 Transformations of the plane

- A *transformation* of the (x, y)-plane is an operation (function) that maps each point of the (x, y)-plane to some other point of the (x, y)-plane.

- If a transformation maps the point P with coordinates (x, y) to the point P' with coordinates (x', y'), then we call P the original point and P' the image point.

- For particular transformations, we are given, or can determine, a formula for x' and y' in terms of x and y.

3.2.2 Example

Find the coordinates of the image points of $A = (2, 1)$ and $B = (-1, 3)$ under the transformation defined by

$$x' = 2x - y \quad \text{and} \quad y' = x^2 + y,$$

and sketch the points and their images.

3.3 Translations

- A *translation* is a transformation that involves shifting each point of the plane a certain distance in the same direction.

- The translation in which each point is shifted a units in the x-direction and b units in the y-direction can be described by the equations $x' = x + a$ and $y' = y + b$.

- This translation can be described in terms of vectors. If $\boldsymbol{p}$ is the position vector of the original point $P = (x, y)$, $\boldsymbol{p}'$ is the position vector of the image point $P' = (x', y')$, and $\boldsymbol{t} = \begin{pmatrix} a \\ b \end{pmatrix}$ is the translation vector, then

$$\boldsymbol{p}' = \boldsymbol{p} + \boldsymbol{t}, \text{ that is, } \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}.$$

- A translation shifts each point the same distance in the same direction, so the shape of figures is preserved.

3.3.1 Example

Draw the triangle with vertices $A = (1, 3)$, $B = (-2, 1)$ and $C = (0, -2)$. Then draw its image under the translation having translation vector $\boldsymbol{t} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$.

3.4 Linear transformations

- A *linear transformation* of the plane is a transformation that maps the point (x, y) to the point (x', y') where

$$x' = ax + by \quad \text{and} \quad y' = cx + dy,$$

  for some real numbers a, b, c and d.

- Such a linear transformation can be represented by the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as follows:

- If $\boldsymbol{p}$ is the position vector of the original point $P = (x, y)$ and $\boldsymbol{p}'$ is the position vector of the image point $P' = (x', y')$, then the linear transformation represented by the matrix A can be described by the equation

$$\boldsymbol{p}' = A\boldsymbol{p}, \text{ that is, } \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

- A linear transformation preserves straight lines. When a linear transformation is applied to a figure, points that are collinear in the original figure map to collinear points in the new figure.

- A linear transformation preserves parallelism. When a linear transformation is applied to a figure, lines that are parallel in the original figure map to lines that are parallel in the new figure.

- A linear transformation maps the origin to itself, since $A\boldsymbol{0} = \boldsymbol{0}$. Note that translations are not linear transformations, except for translatoin by $\boldsymbol{0}$, i.e., the identity transformation.

3.4.1 Example

Are the following transformations linear? If so, write down the matrix representing the transformation, if not, explain why not.

(a) $x' = x + 2y \quad y' = y - x$ (c) $x' = 0 \quad y' = 0$

(b) $x' = x + 2 \quad y' = y - 1$ (d) $x' = 2 \quad y' = -3$

(a) This is linear, represented by the matrix $\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$.

(b) This is not linear. The origin is not mapped to the origin:

$$T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(c) This is linear, represented by the zero matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(d) This is not linear:

$$T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

3.5   Visualizing linear transformations

- The linear transformation represented by the matrix A maps $(1, 0)$ and $(0, 1)$ to points whose position vectors are the first and second columns of A, respectively, since

$$A\boldsymbol{i} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \quad \text{and}$$

$$A\boldsymbol{j} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}.$$

- Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be the position vectors given by the first and second columns of A, respectively.
  Thus $\boldsymbol{a} = \begin{pmatrix} a \\ c \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b \\ d \end{pmatrix}$.

- The linear transformation represented by A will map the point (x, y) to

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = x \begin{pmatrix} a \\ c \end{pmatrix} + y \begin{pmatrix} b \\ d \end{pmatrix}.$$

Thus the point (x, y) is mapped to the point whose position vector is $x\boldsymbol{a} + y\boldsymbol{b} = xA\boldsymbol{i} + yA\boldsymbol{j}$. *The linear transformation will be determined once we know where $\boldsymbol{i}$ and $\boldsymbol{j}$ map to.*

- These facts, along with the knowledge that linear transformations preserve linearity and parallelism, allow us to visualize the effect of a linear transformation by considering the effect of the transformation on the unit grid.

3.6 Scalings

- A scaling of the plane is a stretching (or shrinking) of the plane in two directions, horizontally and vertically.

- A scaling (or dilation) of the plane is a linear transformation.

- The amount of stretching (or shrinking) of a figure under a scaling is determined by two parameters, the scaling factor in the x-direction and the scaling factor in the y-direction. When listing the two scaling factors, we will always list the factor in the x-direction first.

- If one (or both) of the scaling factors is negative, then the image will be reflected in one (or both) of the axes as well as being scaled.

3.6.1 Example

(a) Determine the matrix representing the linear transformation of a scaling that makes a figure twice as wide and three times as high.

> Need $T(\boldsymbol{i})$ and $T(\boldsymbol{j})$:
>
> $$T(\boldsymbol{i}) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}; \quad T(\boldsymbol{j}) = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$
>
> $\Rightarrow$ T is represented by the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

(b) Use the matrix from part a) to determine the image of the point $(2, 3)$ under this scaling.

$$
\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
$$
$$
= \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}.
$$

We now determine the matrix representing a scaling with factor g in the x-direction and factor h in the y-direction.

First we must find the images of the position vectors $\boldsymbol{i}$ and $\boldsymbol{j}$.
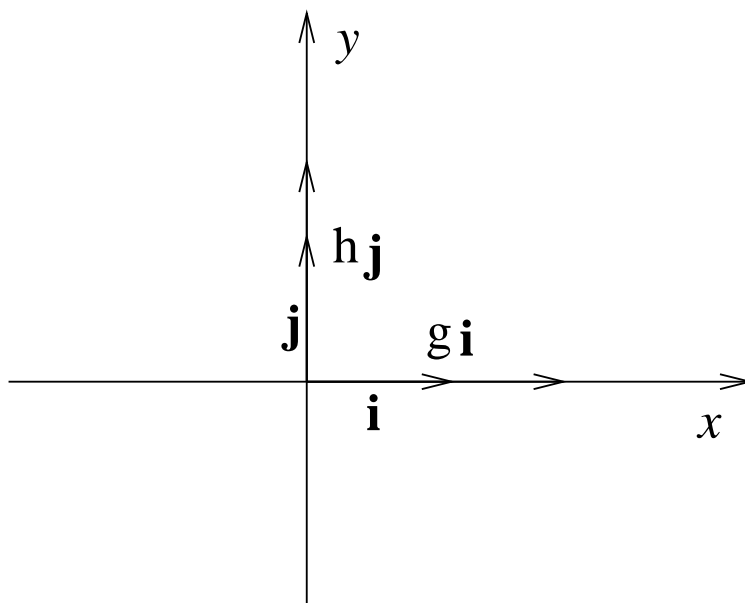
The image of $\boldsymbol{i}$ is the position vector $\begin{pmatrix} g \\ 0 \end{pmatrix}$. The image of $\boldsymbol{j}$ is $\begin{pmatrix} 0 \\ h \end{pmatrix}$.

Thus the matrix representing a scaling with factors g and h is

$$
D_{g,h} = \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}.
$$

3.7 Rotations

- A rotation of the plane about the origin is a linear transformation. A rotation about any other point is not, because the origin will not be mapped to the origin.

- A rotation of the plane through an angle θ will mean an anti-clockwise rotation of the plane about the origin through an angle θ.

3.7.1 Example

(a) Determine the matrix that represents the linear transformation of anti-clockwise rotation through an angle of $\pi/2$ radians about the origin.

$$\mathrm{T}(\boldsymbol{i}) = \boldsymbol{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \ \mathrm{T}(\boldsymbol{j}) = -\boldsymbol{i} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Hence the transform is represented by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(b) Use the matrix from part a) to determine the image of $(2, 3)$ under this rotation.

The point $(2, 3)$ is sent to

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}.$$

We now determine the general matrix that represents an anti-clockwise rotation through an angle θ about the origin.

We must find the images of the position vectors $\boldsymbol{i}$ and $\boldsymbol{j}$. Recall the definition of $\sin\theta$ and $\cos\theta$ in terms of the unit circle.

The image of $\boldsymbol{i}$ is the position vector $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$. The image of $\boldsymbol{j}$ is $\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$.

Thus the matrix representing an anti-clockwise rotation through an angle θ about the origin is

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The diagram shows a unit circle with the point $(\cos\theta, \sin\theta)$ in the first quadrant and the point $\left(\cos\left(\frac{\pi}{2}+\theta\right), \sin\left(\frac{\pi}{2}+\theta\right)\right) = (-\sin\theta, \cos\theta)$ in the second quadrant. Points $(0,1)$, $(1,0)$, vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{i'}$, $\mathbf{j'}$, and angles θ are marked.
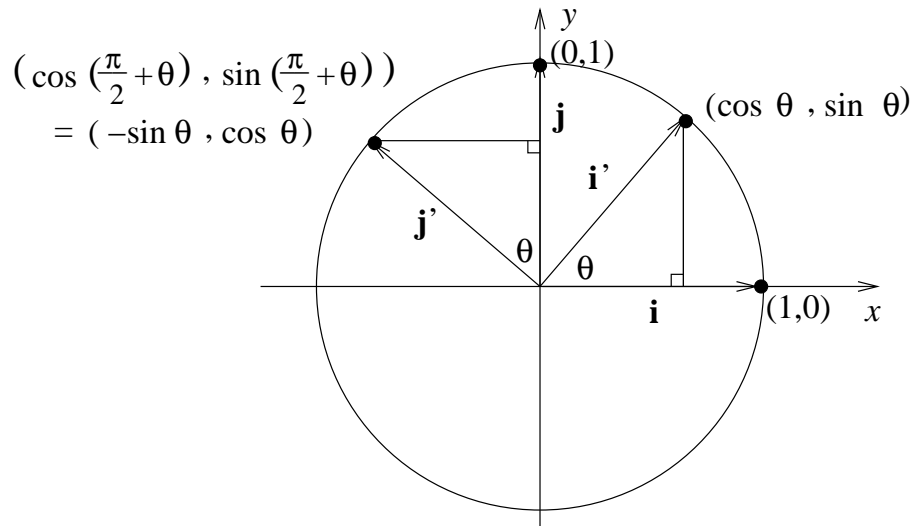
3.7.2   Example

(a) Determine the matrix representing an anti-clockwise rotation through $2\pi/3$ radians about the origin.

> The angle $\theta = 2\pi/3$, so $\cos\theta = -1/2$ and $\sin\theta = \sqrt{3}/2$.
>
> The rotation matrix is $R_{\frac{2\pi}{3}} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$.

(b) Use the matrix from part a) to calculate the image of the point $(2,1)$ under this rotation.

> The image of $(2,1)$ is thus
> $$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 - \frac{\sqrt{3}}{2} \\ \sqrt{3} - \frac{1}{2} \end{pmatrix} \approx \begin{pmatrix} -1.87 \\ 1.23 \end{pmatrix}.$$

3.8   Reflections

- A reflection of the plane in a line through the origin is a linear transformation.

- A reflection in a line that does not pass through the origin does not map the origin to itself, so is not a linear transformation. We will only consider reflections in lines through the origin.

- When the plane is reflected in a line through the origin, points on the line of reflection remain fixed. A point P not on the line of reflection is mapped to a point P' on the opposite side of the line, such that the points P and P' are equidistant from the line of reflection and the line through P and P' is perpendicular to the line of reflection.

3.8.1   Example

Determine the matrix representing the linear transformation of reflection in the x-axis.

$$
T(\boldsymbol{i}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad T(\boldsymbol{j}) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.
$$

Hence $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

We now determine the matrix representing a reflection in the line (passing through the origin) which makes an angle θ with the positive x-axis.

First we must find the images of the position vectors $\boldsymbol{i}$ and $\boldsymbol{j}$. Diagram (a) shows the image of $\boldsymbol{i}$ and diagram (b) shows the image of $\boldsymbol{j}$.
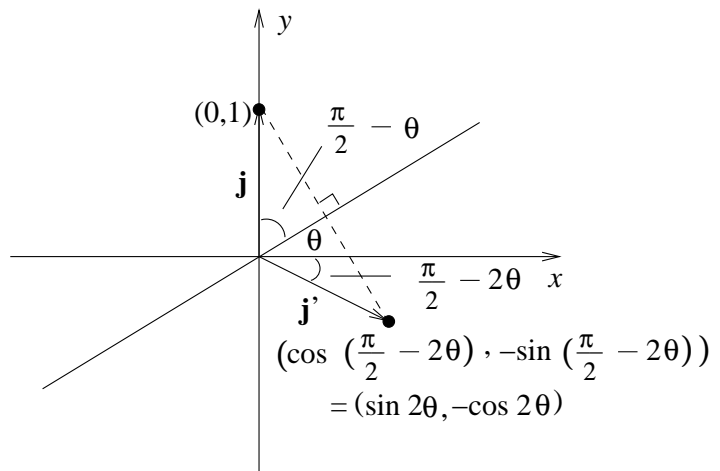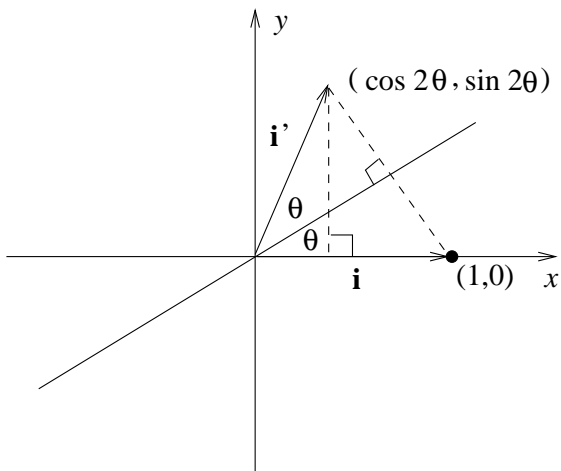
The image of $\boldsymbol{i}$ is the position vector $\begin{pmatrix} \cos 2\theta \\ \sin 2\theta \end{pmatrix}$. The image of $\boldsymbol{j}$ is $\begin{pmatrix} \sin 2\theta \\ -\cos 2\theta \end{pmatrix}$.

Thus the matrix representing a reflection in the line through the origin, that makes an angle θ with the positive x-axis is

$$Q_\theta = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}.$$

3.8.2 Example

(a) Determine the matrix representing a reflection in the line that passes through the origin and through the point $(\sqrt{3}, 1)$.

(b) Use the matrix from part a) to calculate the image of the point $(-1, 1)$ under this reflection.

The line makes an angle of $\theta = \pi/6$ with the x-axis (since $\tan(\pi/6) = 1/\sqrt{3}$). Since $\sin(2\theta) = \sqrt{3}/2$ and $\cos(2\theta) = 1/2$, the reflection matrix is

$$Q_\theta = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.$$

Hence the point $(-1, 1)$ goes to

$$\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{(\sqrt{3}-1)}{2} \\ -\frac{(\sqrt{3}+1)}{2} \end{pmatrix} \approx \begin{pmatrix} 0.37 \\ -1.37 \end{pmatrix}.$$

3.9   Composing Linear Transformations; Matrix Multiplication

Suppose we perform two linear transformations, one after the other. Suppose the first transformation we perform is given by the matrix $B = \begin{pmatrix} s & t \\ u & v \end{pmatrix}$, and the second is given by the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. A vector $\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ is first sent to

$$B\boldsymbol{v} = \begin{pmatrix} s & t \\ u & v \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} sx + ty \\ ux + vy \end{pmatrix}.$$

Then this vector is sent to

$$
\begin{aligned}
A \begin{pmatrix} sx + ty \\ ux + vy \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} sx + ty \\ ux + vy \end{pmatrix} \\
&= \begin{pmatrix} a(sx + ty) + b(ux + vy) \\ c(sx + ty) + d(ux + vy) \end{pmatrix} \\
&= \begin{pmatrix} (as + bu)x + (at + bv)y \\ (cs + du)x + (ct + dv)y \end{pmatrix} = \begin{pmatrix} as + bu & at + bv \\ cs + du & ct + dv \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.
\end{aligned}
$$

We define the *product* AB to be the 2×2 matrix

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} s & t \\ u & v \end{pmatrix} = \begin{pmatrix} as + bu & at + bv \\ cs + du & ct + dv \end{pmatrix}.$$

The effect of performing the transformation represented by B and then the transformation represented by A is to perform the transformation represented by AB. In symbols

$$A(B\boldsymbol{v}) = (AB)\boldsymbol{v}.$$

Mnemonic: we can view B as consisting of two column vectors $\boldsymbol{v} = \begin{pmatrix} s \\ u \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} t \\ v \end{pmatrix}$. Then AB has columns $A\boldsymbol{v}$, $A\boldsymbol{w}$.

3.9.1   Non-commutativity of matrix multiplication

In general $AB \neq BA$. That is, matrix multiplication is *not* commutative.

3.9.2   Example

Let $A = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Then

$$AB = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 0 \cdot 1 + 1 \cdot 3 & 0 \cdot 2 + 1 \cdot 4 \\ 2 \cdot 1 - 1 \cdot 3 & 2 \cdot 2 - 1 \cdot 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ -1 & 0 \end{pmatrix}$$

but

$$BA = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 8 & -1 \end{pmatrix}$$

Note that $AB \neq BA$.

3.10 Vectors in $\mathbb{R}^n$

In a similar way, 3×3 matrices can be used to transform images in 3 dimensional space, by rotating, reflecting etc.

We are familiar with vectors in two and three dimensional space, $\mathbb{R}^2$ and $\mathbb{R}^3$. These generalize to n dimensional space, denoted $\mathbb{R}^n$. A vector $\boldsymbol{v}$ in $\mathbb{R}^n$ is specified by n components:

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \qquad v_i \in \mathbb{R}.$$

We define addition of vectors and multiplication by a scalar α component-wise.

Thus if $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$ is another vector then

$$\boldsymbol{v} + \boldsymbol{w} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{pmatrix} = \boldsymbol{w} + \boldsymbol{v}, \qquad \alpha\boldsymbol{v} = \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \\ \vdots \\ \alpha v_n \end{pmatrix} \qquad \alpha \in \mathbb{R},$$

We may also define the *dot* (or *scalar*) *product* between $\boldsymbol{v}$ and $\boldsymbol{w}$ as

$$\begin{aligned} \boldsymbol{v} \cdot \boldsymbol{w} &= v_1 w_1 + v_2 w_2 + \cdots\cdots + v_n w_n \\ &= \sum_{k=1}^{n} v_k w_k. \end{aligned}$$

The *length* or *norm* of the vector $\boldsymbol{v}$ is

$$
\begin{aligned}
\|\boldsymbol{v}\| &= \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} \\
&= \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}
\end{aligned}
$$

As for $\mathbb{R}^2$ and $\mathbb{R}^3$, every vector $\boldsymbol{v}$ is expressible in terms of the coordinate vectors

$$
\boldsymbol{e}_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{th spot.} \tag{3.1}
$$

Proof:

An arbitrary vector $\boldsymbol{v} \in \mathbb{R}^n$ can be written as

$$
\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} v_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ v_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots\cdots + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ v_n \end{pmatrix}
$$

$$
= v_1 \boldsymbol{e}_1 + v_2 \boldsymbol{e}_2 + \cdots\cdots + v_n \boldsymbol{e}_n.
$$

3.10.1 Notation

(1) E.g. in $\mathbb{R}^3$ we have

$$
\boldsymbol{i} = \boldsymbol{e}_1, \quad \boldsymbol{j} = \boldsymbol{e}_2, \quad \boldsymbol{k} = \boldsymbol{e}_3.
$$

(2) We have also the zero vector

$$
\boldsymbol{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \text{satisfying}
$$

$$
\boldsymbol{0} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{0} = \boldsymbol{v}, \qquad 0 \cdot \boldsymbol{v} = \boldsymbol{0}, \qquad \text{for every vector } \boldsymbol{v}.
$$

(3) If $\boldsymbol{v} \neq \boldsymbol{0}$, $\boldsymbol{w} \neq \boldsymbol{0}$, we say that vectors $\boldsymbol{v}$, $\boldsymbol{w}$ are *perpendicular* (or *orthogonal*) if
$$\boldsymbol{v} \cdot \boldsymbol{w} = 0.$$

3.11 Properties of Dot Product

If $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ then

(1) $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$.

(2) $\boldsymbol{v} \cdot (\boldsymbol{w} + \boldsymbol{u}) = \boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \boldsymbol{u}$.

(3) $(\alpha\boldsymbol{v}) \cdot \boldsymbol{w} = \alpha(\boldsymbol{v} \cdot \boldsymbol{w}) = \boldsymbol{v} \cdot (\alpha\boldsymbol{w})$.

(4) $\boldsymbol{v} \cdot \boldsymbol{v} \geq 0$ and $\boldsymbol{v} \cdot \boldsymbol{v} = 0$ if and only if $\boldsymbol{v} = \boldsymbol{0}$.

3.12 Definition: Matrix

A rectangular array of numbers
$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{pmatrix}$$
is called an $m \times n$ matrix – made up of m rows and n columns. Entries of the jth column may be assembled into a vector
$$\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$$
called the jth *column vector* of A. Similarly the ith *row vector* of A is
$$\begin{pmatrix} a_{i1} \, a_{i2} \cdots\cdots a_{in} \end{pmatrix}.$$

Note that an $m \times 1$ matrix is a column vector and a $1 \times n$ matrix is a row vector. The number a_{ij} in the ith row and j column of A is called the (i, j)th *entry* of A. For brevity we write $A = (a_{ij})$.

3.12.1 Example

The 2×3 matrix with entries $a_{ij} = i - j$ is
$$A = \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \end{pmatrix}.$$

3.12.2 Example

The $m \times n$ matrix A whose entries are all zero is called the *zero matrix*, denoted O; eg. the zero 3×2 matrix is

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

3.13 Equality

Matrices A, B are said to be *equal*, written $A = B$, if they are the same size and $a_{ij} = b_{ij}$, for all i, j.

3.14 Addition

The *sum* of two $m \times n$ matrices A and B is defined to be the $m \times n$ matrix $A + B$ with entries

$$(A + B)_{ij} = a_{ij} + b_{ij}$$

We add matrices element wise, as for vectors.

Addition is only defined between matrices of the same size.

3.14.1 Properties

For matrices A, B, C of the same size we have

$$\begin{aligned} A + B &= B + A \\ (A + B) + C &= A + (B + C). \end{aligned}$$

3.15 Scalar Multiplication

Let A be an $m \times n$ matrix and $\alpha \in \mathbb{R}$. We define αA to be the $m \times n$ matrix with entries

$$(\alpha A)_{ij} = \alpha \cdot a_{ij} \quad , \quad \text{for all } i, j.$$

Throughout we write $-A$ for $(-1) \cdot A$. We may then define subtraction between matrices of the same size by

$$A - B = A + (-B).$$

3.15.1 Example

Let

$$A = \begin{pmatrix} -4 & 6 & 3 \\ 0 & 1 & 2 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \end{pmatrix}.$$

Then

$$A + B = \begin{pmatrix} 1 & 5 & 3 \\ 3 & 2 & 2 \end{pmatrix} \quad , \quad A - B = \begin{pmatrix} -9 & 7 & 3 \\ -3 & 0 & 2 \end{pmatrix}.$$

3.15.2 Example

If $A = \begin{pmatrix} 6 & 0 \\ 2 & -1 \end{pmatrix}$ then $A + A = \begin{pmatrix} 12 & 0 \\ 4 & -2 \end{pmatrix} = 2A.$

3.16 Matrix Multiplication

Let $A = (a_{ij})$ be an $m \times n$ and $B = (b_{jk})$ an $n \times r$ matrix. Then AB is the $m \times r$ matrix with ik entry

$$(ab)_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots\cdots + a_{in}b_{nk} :$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots\cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & \cdots\cdot & b_{1k} & \cdots\cdot & b_{1r} \\ b_{21} & \cdots\cdot & b_{2k} & \cdots\cdot & b_{2r} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ b_{n1} & \cdots\cdot & b_{nk} & \cdots\cdot & b_{nr} \end{pmatrix} = \begin{matrix} AB \\ (m \times r) \end{matrix}$$

$$\begin{matrix} A \\ (m \times n) \end{matrix} \qquad\qquad\qquad\qquad \begin{matrix} B \\ (n \times r) \end{matrix}$$

where $(ab)_{ik} =$ dot product between ith row of A and kth column of B.

AB is only defined if the number of columns $A =$ the number of rows of B.

3.16.1 Example

- Let $A = (-1\ 0\ 1)$ and $B = \begin{pmatrix} 0 & -1 \\ 1 & 2 \\ -2 & \frac{1}{2} \end{pmatrix}$
  1×3 3×2

- Then AB is defined, but $\underset{3\times2}{B} \underset{1\times3}{A}$ is not defined.

Calculate AB.

$$AB = (-1\ 0\ 1) \begin{pmatrix} 0 & -1 \\ 1 & 2 \\ -2 & \frac{1}{2} \end{pmatrix}$$

$$= \left(-1 \cdot 0 + 0 \cdot 1 + 1 \cdot -2 \quad\quad -1 \cdot 1 + 0 \cdot 2 + 1 \cdot \frac{1}{2} \right)$$

$$= (-2\ \ 3/2)$$

3.16.2   Example

$$A = (1\ 3\ 9),\ \ B = \begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix}.$$

Calculate AB and BA.

$$\underset{1\times3}{(1\ 3\ 9)} \underset{3\times1}{\begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix}} = 2 + 3 + 63 = 68$$

$$\underset{3\times1}{\begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix}} \underset{1\times3}{(1\ 3\ 9)} = \begin{pmatrix} 2 & 6 & 18 \\ 1 & 3 & 9 \\ 7 & 21 & 63 \end{pmatrix}.$$

3.16.3   Properties

For matrices of appropriate size

(1) $(AB)C = A(BC)$,   Associativity

(2) $(A+B)C = AC + BC$, $A(B+C) = AB + AC$ Distributive Laws

Another unusual property of matrices is that $AB = 0$ does *not* imply $A = 0$ or $B = 0$. It is possible for the product of two non-zero matrices to be zero:

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

40

Also, if $AC = BC$, or $CA = CB$ then it is not true in general that $A = B$. (If $AC = BC$, then $AC - BC = 0$ and $(A - B)C = 0$, but this does **not** imply $A - B = 0$ or $C = 0$.)

3.17 Transposition

The *transpose* of an $m \times n$ matrix $A = (a_{ij})$ is the $n \times m$ matrix A^T with entries

$$a_{ji} = a_{ij}^T \quad , \quad \text{for all } i, \, j$$

ie.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} & \cdots\cdots & a_{m1} \\ a_{12} & a_{22} & \cdots\cdots & a_{m2} \\ \vdots & & & \vdots \\ a_{1n} & a_{2n} & \cdots\cdots & a_{mn} \end{pmatrix}$$

So the row vectors of A become column vectors of A^T and vice versa.

3.17.1 Examples

$$\begin{pmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{pmatrix}^T = \begin{pmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$$

$$(7 \ \ 5 \ \ -2)^T = \begin{pmatrix} 7 \\ 5 \\ -2 \end{pmatrix}$$

3.17.2 Properties

For matrices of appropriate size

(1) $(\alpha A)^T = \alpha \cdot A^T \quad , \quad \alpha \in \mathbb{R}$

(2) $(A + B)^T = A^T + B^T$

(3) $(A^T)^T = A$

(4) $(AB)^T = B^T A^T \qquad (not \ A^T B^T!).$

3.17.3   Dot product expressed as matrix multiplication

A column vector

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$$

may be interpreted as an $n \times 1$ matrix. Then the dot product (for two column vectors $\boldsymbol{v}$ and $\boldsymbol{w}$) may be expressed using matrix multiplication:

$$
\begin{aligned}
\boldsymbol{v} \cdot \boldsymbol{w} &= v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = \underset{1 \times n}{\begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}} \underset{n \times 1}{\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}} \\
&= \boldsymbol{v}^T \boldsymbol{w}.
\end{aligned}
$$

Also

$$
\begin{aligned}
\boldsymbol{v} \cdot \boldsymbol{v} &= v_1^2 + v_2^2 + \cdots + v_n^2 = \|v\|^2 \\
&= \boldsymbol{v}^T \boldsymbol{v}.
\end{aligned}
$$

4 Vector Spaces

In this chapter we consider sets of vectors with special properties, called vector spaces.

4.1 Linear combinations

4.1.1 Definition: Linear combination

If $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_m \in \mathbb{R}^n$, a vector of the form

$$\boldsymbol{v} = \alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_m \boldsymbol{v}_m \quad , \quad \alpha_i \in \mathbb{R}$$

is called a *linear combination* of the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_m$.

4.1.2 Example

Every $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ is a linear combination of the coordinate vectors e_i:

$$\boldsymbol{v} = v_1 \boldsymbol{e}_1 + v_2 \boldsymbol{e}_2 + \cdots + v_n \boldsymbol{e}_n.$$

4.1.3 Example

Show that each of the vectors

$$\boldsymbol{w}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \boldsymbol{w}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ \text{and} \ \boldsymbol{w}_3 = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

are a linear combination of

$$\boldsymbol{v}_1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \ \text{and} \ \boldsymbol{v}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Write

$$\boldsymbol{w}_1 = \alpha_1 \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Equating components:

$$\begin{aligned} 2\alpha_1 + 3\alpha_2 &= 1, \\ 2\alpha_1 + 2\alpha_2 &= 0 \end{aligned}$$

Solving, $\alpha_2 = 1$ and $\alpha_1 = -1$. Similarly we get:

$$
\begin{aligned}
\boldsymbol{w}_1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (-1)\begin{pmatrix} 2 \\ 2 \end{pmatrix} + 1\begin{pmatrix} 3 \\ 2 \end{pmatrix} \\[2mm]
&= -\boldsymbol{v}_1 + \boldsymbol{v}_2, \\[4mm]
\boldsymbol{w}_2 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{3}{2}\begin{pmatrix} 2 \\ 2 \end{pmatrix} - 1\begin{pmatrix} 3 \\ 2 \end{pmatrix} \\[2mm]
&= \frac{3}{2}\boldsymbol{v}_1 - \boldsymbol{v}_2, \\[4mm]
\boldsymbol{w}_3 &= \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \frac{3}{2}\begin{pmatrix} 2 \\ 2 \end{pmatrix} + 0\begin{pmatrix} 3 \\ 3 \end{pmatrix} \\[2mm]
&= \frac{3}{2}\boldsymbol{v}_1 + 0\boldsymbol{v}_2.
\end{aligned}
$$

4.2 Linear Independence

4.2.1 Definition: Linear independence

Consider the linear combination

$$\alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2 + \cdots + \alpha_m\boldsymbol{v}_m$$

with $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Obviously this gives the 0 vector.

A set of vectors $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_m\} \subseteq \mathbb{R}^n$ is *linearly dependent* if there exist scalars $\alpha_1, ..., \alpha_m$ *not all zero* such that

$$\alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2 + \cdots + \alpha_m\boldsymbol{v}_m = \boldsymbol{0}.$$

S is called *linearly independent* if no such scalars exist, i.e. if the only linear combination adding up to $\boldsymbol{0}$ is with all the scalars 0. So S is linearly independent if

$$\alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2 + \cdots + \alpha_m\boldsymbol{v}_m = \boldsymbol{0} \quad \Rightarrow \quad \alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$$

4.3 How to test for linear independence

Given vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$, solve the vector equation

$$\alpha_1\boldsymbol{v}_1 + \cdots + \alpha_m\boldsymbol{v}_m = \boldsymbol{0}.$$

If the *only* solution is $\alpha_1, \ldots, \alpha_m = 0$ then the vectors are linearly independent.

If there is any other solution, the vectors are linearly dependent.

4.3.1 Example

Show that in $\mathbb{R}^2$, the vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are linearly independent.

$$\alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \alpha + \beta \\ \alpha - \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left. \begin{array}{c} \alpha + \beta = 0 \\ \alpha - \beta = 0 \end{array} \right\} \Rightarrow \alpha = \beta = 0.$$

4.3.2 Example

Show that in $\mathbb{R}^n$ the coordinate vectors $\boldsymbol{e}_i$ are linearly independent.

$$\alpha_1 \boldsymbol{e}_1 + \alpha_2 \boldsymbol{e}_2 + \cdots + \alpha_n \boldsymbol{e}_n = \boldsymbol{0}$$

$$\Rightarrow \begin{pmatrix} \alpha_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \alpha_2 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

$$\Rightarrow \alpha_1 = 0, \ \alpha_2 = 0, \ \ldots, \ \alpha_n = 0.$$

4.3.3 Example

Show that in $\mathbb{R}^3$, the vectors $\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ are linearly dependent.

$$\alpha_1 \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} \alpha_1 + \alpha_2 & = 0 \\ 3\alpha_1 + \alpha_2 + \alpha_3 & = 0 \\ \alpha_1 + \alpha_2 & = 0 \end{cases}$$

Note that the first and third equations are the same, so we effectively only have 2 equations in 3 unknowns.

$\alpha_2 = -\alpha_1$ and $\alpha_3 = -3\alpha_1 - \alpha_2 = -2\alpha_1$. α_1 is not determined: it can be anything. For example if we choose $\alpha_1 = 1$ then $\alpha_2 = -1$ and $\alpha_3 = -2$.

Check:

$$\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We have found a non-trivial linear combination summing to $\mathbf{0}$, so the vectors are linearly dependent.

4.3.4 Example

Show that any finite set of vectors containing the zero vector $\mathbf{0}$ is linearly dependent.

Write $S = \{\boldsymbol{v}_1, ..., \boldsymbol{v}_m, \mathbf{0}\} \subseteq \mathbb{R}^n$. Then

$$0 \cdot \boldsymbol{v}_1 + \cdots + 0 \cdot \boldsymbol{v}_m + 1 \cdot \mathbf{0} = \mathbf{0}$$

$\Rightarrow$ S linearly dependent.

Special cases:

(i) If **two** vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ are linearly dependent, then there exist scalars α_1, $\alpha_2 \in \mathbb{R}$, not both 0, with

$$\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 = \mathbf{0}.$$

If $\alpha_2 \neq 0$ then $\boldsymbol{v}_2 = (-\alpha_1/\alpha_2)\boldsymbol{v}_1$, so $\boldsymbol{v}_2$ is a scalar multiple of $\boldsymbol{v}_1$. If $\alpha_2 = 0$ then $\alpha_1 \neq 0$ so $\boldsymbol{v}_1 = \mathbf{0} = 0\boldsymbol{v}_2$, and $\boldsymbol{v}_2$ is a scalar multiple of $\boldsymbol{v}_2$.

So two vectors are linearly dependent if and only if one of them is a multiple of the other.

(ii) If **one** vector $\boldsymbol{v}$ is linearly dependent then $\alpha \boldsymbol{v} = \mathbf{0}$ with $\alpha \neq 0$, so $\boldsymbol{v} = \mathbf{0}$. Thus a single vector is linearly dependent if and only if it is the zero vector.

4.4 Vector spaces

4.4.1 Definition: Vector space, subspace

A subset $V \subseteq \mathbb{R}^n$ is called a *vector space* if it satisfies the following

(o) V is non-empty.

(i) If $\boldsymbol{v} \in V$ and $\boldsymbol{w} \in V$ then $\boldsymbol{v} + \boldsymbol{w} \in V$ (closure under vector addition).

(ii) If $\boldsymbol{v} \in V$ and $\alpha \in \mathbb{R}$ then $\alpha\boldsymbol{v} \in V$ (closure under scalar multiplication).

If V is a vector space, $W \subseteq V$ and W is also a vector space, we call W a *subspace* of V. Note: V is a subspace of itself.

If $V \subseteq \mathbb{R}^n$ is a vector space and $\boldsymbol{v} \in V$, then by property (ii)

$$\boldsymbol{0} = 0\boldsymbol{v} \in V$$

that is, the zero vector belongs to every vector space.

4.4.2 Example

Show that

$$V = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \,\middle|\, a + b = 2c \right\}, \quad a, b, c \in \mathbb{R}$$

is a vector space.

(o) V is non-empty since $\boldsymbol{0} \in V$.

Let $\boldsymbol{v} = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, \quad \boldsymbol{w} = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$ be in V.

(i) $\boldsymbol{v} + \boldsymbol{w} = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \end{pmatrix}$

$$\boldsymbol{v} \in V \Rightarrow a_1 + b_1 = 2c_1$$
$$\boldsymbol{w} \in V \Rightarrow a_2 + b_2 = 2c_2.$$

Is $\boldsymbol{v} + \boldsymbol{w}$ in V?

$$(a_1 + a_2) + (b_1 + b_2) = (a_1 + b_1) + (a_2 + b_2)$$
$$= 2c_1 + 2c_2 = 2(c_1 + c_2)$$

$\Rightarrow$ $\boldsymbol{v} + \boldsymbol{w} \in V$. So V is closed under addition.

Consider $\boldsymbol{v} = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}$, $\quad \boldsymbol{w} = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \in V$.

(ii) If α is any scalar then $\alpha \boldsymbol{v} = \begin{pmatrix} \alpha a_1 \\ \alpha b_1 \\ \alpha c_1 \end{pmatrix}$

$\alpha a_1 + \alpha b_1 = \alpha(a_1 + b_1) = \alpha 2 c_1 = 2(\alpha c_1)$

$\Rightarrow \alpha \boldsymbol{v} \in V$. So V is closed under scalar multiplication.

Hence V is a vector space.

4.4.3 Example

Is $V = \{\boldsymbol{0}\}$ a vector space?

(o) $\boldsymbol{0} \in V \Rightarrow V$ is non-empty.

(i) If $\boldsymbol{v}, \boldsymbol{w} \in V$ then $\boldsymbol{v} = \boldsymbol{w} = \boldsymbol{0}$ and $\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{0} \in V$.

(ii) If $\alpha \in \mathbb{R}$, then $\alpha \boldsymbol{0} = \boldsymbol{0} \in V$.

4.4.4 Example

Show that $V = \mathbb{R}^n$ is a vector space.

(o) $\boldsymbol{0} \in V$

(i) $\boldsymbol{v}, \boldsymbol{w} \in V \Rightarrow \boldsymbol{v} + \boldsymbol{w} \in \mathbb{R}^n = V$

(ii) $\boldsymbol{v} \in V \Rightarrow \alpha \boldsymbol{v} \in \mathbb{R}^n = V$

So V is a vector space.

4.4.5 Example

Is $V = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$ a vector space?

No, since $\mathbf{0} \notin V$.

4.4.6 Example

Is $V = \left\{ \alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mid \alpha \in \mathbb{R} \right\}$ a vector space?

(o) $\alpha = 0 \Rightarrow \mathbf{0} \in V$.

(i) If $\mathbf{v}, \mathbf{w} \in V$ then $\mathbf{v} = \alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{w} = \beta \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ for suitable $\alpha, \beta \in \mathbb{R}$.

$$
\begin{aligned}
\text{Then} \qquad \mathbf{v} + \mathbf{w} &= \begin{pmatrix} \alpha \\ 2\alpha \end{pmatrix} + \begin{pmatrix} \beta \\ 2\beta \end{pmatrix} \\
&= \begin{pmatrix} \alpha + \beta \\ 2(\alpha + \beta) \end{pmatrix} \\
&= (\alpha + \beta) \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in V.
\end{aligned}
$$

(ii) If $\gamma \in \mathbb{R}$, then $\gamma \mathbf{v} = \gamma \alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in V$.

Therefore V is a vector space.

Geometrically, all these vectors can be viewed as lying along a single line through the origin.

4.4.7 Example

Show that for $\boldsymbol{v}_1$, $\boldsymbol{v}_2 \in \mathbb{R}^3$,

$$V = \{\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 \mid \alpha_1,\, \alpha_2 \in \mathbb{R}\}$$

is a vector space.

(o) If we choose $\alpha_1 = \alpha_2 = 0$ we see that $0\boldsymbol{v}_1 + 0\boldsymbol{v}_2 = \boldsymbol{0} \in V$, so V is non-empty.

(i) Let $\boldsymbol{v}, \boldsymbol{w} \in V \Rightarrow \exists\ \alpha_1, \alpha_2, \beta_1, \beta_2$ such that $\boldsymbol{v} = \alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2$ and $\boldsymbol{w} = \beta_1 \boldsymbol{v}_1 + \beta_2 \boldsymbol{v}_2$. Then $\boldsymbol{v} + \boldsymbol{w} = (\alpha_1 + \beta_1)\boldsymbol{v}_1 + (\alpha_2 + \beta_2)\boldsymbol{v}_2 \in V$.

(ii) For any scalar $\gamma \in \mathbb{R}$, $\gamma \boldsymbol{v} = \gamma\alpha_1 \boldsymbol{v}_1 + \gamma\alpha_2 \boldsymbol{v}_2 \in V$.

Therefore V is a vector space.

If we draw $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ starting from the origin, then all these combinations $\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2$ will lie in a single plane in $\mathbb{R}^3$, through the origin.

For example, if $\boldsymbol{v}_1 = \boldsymbol{i}$ and $\boldsymbol{v}_2 = \boldsymbol{j}$ then $V = \{\alpha_1 \boldsymbol{i} + \alpha_2 \boldsymbol{j} \mid \alpha_1, \alpha_2 \in \mathbb{R}\}$ is the set of vectors with no k-component i.e. the xy plane.

Thus geometrically, a vector space can be a single point $\{0\}$, or can be all vectors along a single line through the origin, or all vectors lying in a single plane through the origin or similarly in higher dimensions.

4.5 The span of vectors

We would like a compact way to describe all the vectors in a vector space V, and a way of generating a vector space from a given set of vectors.

4.5.1 Definition: Span

The *span* of vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, ..., $\boldsymbol{v}_m \in \mathbb{R}^n$ is the set of all possible linear combinations of these vectors:

$$\mathrm{span}\{\boldsymbol{v}_1,\, \boldsymbol{v}_2,\, ...,\, \boldsymbol{v}_m\} = \{\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_m \boldsymbol{v}_m \mid \alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}\}.$$

Note that the span is a *set* of vectors.

4.5.2 The Span of vectors is a Vector Space

The span of any non-empty set of vectors is a vector space.

We proved this for two vectors in Example 4.4.7. The general proof is similar.

If $\text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_m\} = V$ then we say that $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ *span* V, or that they form a *spanning set* for V.

4.5.3 Example

Find a spanning set for $V = \left\{ \alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mid \alpha \in \mathbb{R} \right\}$.

Every vector in V is a linear combination of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, so a spanning set is $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$.

This answer is not unique. For example $\left\{ \begin{pmatrix} -1 \\ -2 \end{pmatrix} \right\}$ is another spanning set for V.

4.5.4 Example

The coordinate vectors $\boldsymbol{e}_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i \qquad$ span $\mathbb{R}^n$ since every vector $\boldsymbol{v} \in \mathbb{R}^n$

is a linear combination of these vectors (see equation 3.1).

4.5.5 Example

The vector space $\{\boldsymbol{0}\}$ is spanned by the zero vector $\boldsymbol{0} \in \mathbb{R}^n$ is and called the *zero vector space*.

4.6   Bases

We have seen that we can describe a vector space by giving a spanning set. When we do this, we usually like to choose a spanning set that is as small as possible (contains no redundancies). We ensure this as follows.

4.6.1   Definition: Basis

Let V be a vector space. We say that a set of vectors $B = \{v_1, ..., v_m\} \subseteq V$ is a *basis* for V if:

1. B spans V;

2. B is linearly independent.

To prove that a set of vectors is a basis, we must prove both properties.

Sometimes we are just given a vector space V, and asked to find a basis. Problems like this are solved by finding a spanning set for V, and then checking that the set is linearly independent.

A single vector space V may have many different bases.

4.6.2   Example

We have seen that the coordinate vectors e_i, $i = 1, ..., n$, are linearly independent (section 4.3.2) and span $\mathbb{R}^n$ (section 4.5.4). Therefore they form a basis for $\mathbb{R}^n$, called the *standard basis*.

4.6.3   Example

Show that the vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ form a basis for $\mathbb{R}^2$.

Need to check: (i) linear independence; (ii) the set spans $\mathbb{R}^2$.

(i) We have already seen that these vectors are linearly independent (Example 4.3.1).

(ii) Let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2$ be arbitrary. We need to find α_1 and $\alpha_2 \in \mathbb{R}$ s.t.

$$v = \alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \alpha_1 + \alpha_2 \\ \alpha_1 - \alpha_2 \end{pmatrix}.$$

So we need:
$$\begin{cases} \boldsymbol{v}_1 = \alpha_1 + \alpha_2 \\ \boldsymbol{v}_2 = \alpha_1 - \alpha_2 \end{cases} \Rightarrow \begin{cases} \alpha_1 = \dfrac{1}{2}(v_1 + v_2) \\ \alpha_2 = \dfrac{1}{2}(v_1 - v_2) \end{cases}.$$

Hence
$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{2}(v_1 + v_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}(v_1 - v_2) \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Note that this basis differs from the standard $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ basis of $\mathbb{R}^2$, but still has two elements.

4.6.4 Example

Find a basis for the vector space of 4.4.2

$$V = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \,\middle|\, a + b = 2c \right\}, \quad a, b, c \in \mathbb{R}$$

Any vector in V is of the form

$$\boldsymbol{v} = \begin{pmatrix} 2c - b \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2c \\ 0 \\ c \end{pmatrix} + \begin{pmatrix} -b \\ b \\ 0 \end{pmatrix} = c\boldsymbol{v}_1 + b\boldsymbol{v}_2$$

with $c, b \in \mathbb{R}$. Thus $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is a spanning set for V. For $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ to be a basis, we need to show $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly independent. If $\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 = \boldsymbol{0}$ then from the second coordinate $\alpha_2 = 0$ and from the first $\alpha_1 = 0$, so $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is also linearly independent.

4.7 Dimension

Suppose $B = \{v_1, ..., v_m\}$ is a basis for V. Then it can be shown that *every* basis for V has m vectors. We call m the *dimension* of V and write $m = \dim V$. (By convention, the zero vector space $\{0\}$ has dimension zero.)

4.7.1 Example

We have seen that the coordinate vectors e_i, form a basis for $\mathbb{R}^n$, so

$$\dim \mathbb{R}^n = n.$$

4.7.2 Example

The vector space

$$V = \left\{ \alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mid \alpha \in \mathbb{R} \right\} \subseteq \mathbb{R}^2$$

is spanned by a single vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ (example 4.5.3) which thus constitutes a basis for V. Hence $\dim V = 1$.

Geometrically, V consists of vectors along a single line, which is a one dimensional object.

4.7.3 Example

Consider the vector space of 4.4.2

$$V = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \;\middle|\; a + b = 2c \right\}, \quad a, b, c \in \mathbb{R}$$

In example 4.6.4 we found a basis with 2 vectors, so $\dim V = 2$. Note that V is a subspace of $\mathbb{R}^3$, but it consists of a plane through the origin, which is indeed a two dimensional object.

4.8   Components

Suppose $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$ is a basis for V. Then every $\boldsymbol{v} \in V$ may be written as a linear combination of elements in B

$$\boldsymbol{v} = \alpha_1 \boldsymbol{v}_1 + \cdots \cdots + \alpha_m \boldsymbol{v}_m$$

and this representation is *unique*: if we also have

$$\boldsymbol{v} = \beta_1 \boldsymbol{v}_1 + \cdots \cdots + \beta_m \boldsymbol{v}_m$$

then $\alpha_1 = \beta_1, \ldots, \alpha_m = \beta_m$.

Proof:

Since B is a basis, it spans V.

Hence every $\boldsymbol{v}$ can be written as some linear combination of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$.

We need to prove uniqueness. But

$$\begin{aligned} \boldsymbol{0} &= \boldsymbol{v} - \boldsymbol{v} = (\alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_m \boldsymbol{v}_m) - (\beta_1 \boldsymbol{v}_1 + \cdots + \beta_m \boldsymbol{v}_m) \\ &= (\alpha_1 - \beta_1)\boldsymbol{v}_1 + \cdots + (\alpha_m - \beta_m)\boldsymbol{v}_m. \end{aligned}$$

Since B is linearly independent, $\alpha_i - \beta_i = 0$, $i = 1, \ldots, m$.

So $\alpha_i = \beta_i$ meaning that the representation is unique.

If $\boldsymbol{v} = \alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_m \boldsymbol{v}_m$ we call α_i the ith *component* of $\boldsymbol{v}$ with respect to the basis B.

4.8.1   Example

For $\boldsymbol{v} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, find the components of $\boldsymbol{v}$ with respect to the basis $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$.

We want $\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Solving, $\alpha = 4$, $\beta = -1$ so the components are $4, -1$.

4.8.2 Example

Write $\begin{pmatrix} 7 \\ 3 \\ 1 \end{pmatrix}$ with respect to the basis $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$

We have to write

$$\begin{pmatrix} 7 \\ 3 \\ 1 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

This leads to equations

$$\begin{array}{rrrrl} \alpha & + \ \beta & + \ \gamma & = & 7 \\ & \beta & + \ \gamma & = & 3 \\ & & \gamma & = & 1 \end{array}$$

Solving, we get $\alpha = 4$, $\beta = 2$, $\gamma = 1$, so

$$\begin{pmatrix} 7 \\ 3 \\ 1 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

This is the **unique** way to write this vector with respect to this basis.

4.9 Further Properties of Bases

With more work, we can also prove the following:

(i) Every vector space has a basis.

(ii) If $m = \dim V$, *any* m linearly independent vectors in V will form a basis.

(iii) If $m = \dim V$, any set of more than m vectors in V will be linearly dependent (and so not be a basis).

(iv) If $m = \dim V$, any set of fewer than m vectors in V will not span V, (and so not be a basis).

(v) If W is a subspace of V, then $\dim W \leq \dim V$ and $\dim W = \dim V$ iff $V = W$.

(vi) The only m-dimensional subspace of $\mathbb{R}^m$ is $\mathbb{R}^m$ itself.

4.9.1 Example

Show that in $\mathbb{R}^3$, the vectors

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \boldsymbol{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

are linearly independent and hence form a basis for $\mathbb{R}^3$.

Setting $\alpha \boldsymbol{v}_1 + \beta \boldsymbol{v}_2 + \gamma \boldsymbol{v}_3 = \boldsymbol{0}$ implies

$$\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \alpha + \beta + \gamma \\ \beta + \gamma \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \gamma = 0 \Rightarrow \beta = 0 \Rightarrow \alpha = 0$$

Therefore the vectors are linearly independent.
Because we already know $\dim \mathbb{R}^3 = 3$ and we have found 3 linearly independent vectors, they must form a basis, by (vi) above.

Check: in $\mathbb{R}^3$, $\boldsymbol{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\Rightarrow \boldsymbol{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = (a-b) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (b-c) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The shortcut in this example is fine if we already know the dimension of V. If we are just given a space like

$$V = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \,\Big|\, a + b = 2c \right\}, \ a, b, c \in \mathbb{R}$$

it is often not obvious what the dimension is (2? 3?).

4.10   Orthogonal and Orthonormal

As we have seen, a single vector space can have many different bases. One reason we like the standard basis $\{\boldsymbol{e}_i\}$ of $\mathbb{R}^n$ is that the basis vectors are mutually orthogonal (perpendicular). The *Gram-Schmidt process* is a method of using a given basis for a vector space to obtain a new basis in which the basis vectors are orthogonal.

4.10.1 Definition: Orthogonal set, orthonormal set

A set S of vectors is *orthogonal* or *mutually orthogonal* if each pair of distinct vectors in S are orthogonal. It is *orthonormal* if additionally, all the vectors have norm 1.

Note: in an orthogonal set all the vectors must be non-zero, because the 0-vector is not considered to be orthogonal to other vectors.

In symbols, $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is *orthogonal* if

$$\boldsymbol{v}_i \cdot \boldsymbol{v}_j = 0 \qquad \text{for all } i \neq j$$

It is *orthonormal* if also

$$\boldsymbol{v}_i \cdot \boldsymbol{v}_i = 1 \qquad \text{for all } i$$

4.10.2 Example

The standard basis $\{\boldsymbol{i}, \boldsymbol{j}\}$ in $\mathbb{R}^2$ is an orthonormal set, because each vector has norm 1, and $\boldsymbol{i} \cdot \boldsymbol{j} = 0$. More generally the standard basis $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ of $\mathbb{R}^n$ is an orthonormal set.

4.10.3 Example

The set

$$\left\{ \boldsymbol{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

is an orthogonal set in $\mathbb{R}^3$ but is not orthonormal. We have $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = \boldsymbol{v}_2 \cdot \boldsymbol{v}_3 = \boldsymbol{v}_3 \cdot \boldsymbol{v}_1 = 0$, but $\|\boldsymbol{v}_2\| = \sqrt{2}, \|\boldsymbol{v}_3\| = \sqrt{2}$.

4.10.4 Converting Orthogonal to Orthonormal

It is easy to convert an orthogonal set into an orthonormal one: just make each vector into a unit vector, by dividing by its norm. (This does not change the angle between the vectors, so the set is still orthogonal.)

4.10.5 Example

The set

$$\left\{ \boldsymbol{u}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{u}_2 = \frac{1}{\sqrt{2}}\boldsymbol{v}_2 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}, \quad \boldsymbol{u}_3 = \frac{1}{\sqrt{2}}\boldsymbol{v}_3 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix} \right\}$$

is an orthonormal set made from the orthogonal set in the previous example.

4.10.6 Orthogonal implies Linearly Independent

Any orthogonal set of vectors is linearly independent.

Proof: Let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ be an orthogonal set. Suppose

$$\alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_n \boldsymbol{v}_n = \boldsymbol{0}. \tag{4.1}$$

We must show $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Take the dot product of equation 4.1 with $\boldsymbol{v}_1$:

$$
\begin{aligned}
0 &= (\alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_n \boldsymbol{v}_n) \cdot \boldsymbol{v}_1 \\
&= \alpha_1 (\boldsymbol{v}_1 \cdot \boldsymbol{v}_1) + \alpha_2 (\boldsymbol{v}_2 \cdot \boldsymbol{v}_1) + \cdots + \alpha_n (\boldsymbol{v}_n \cdot \boldsymbol{v}_1) \\
&= \alpha_1 \|\boldsymbol{v}_1\|^2
\end{aligned}
$$

where we used orthogonality: $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 0$, $\boldsymbol{v}_1 \cdot \boldsymbol{v}_3 = 0$ etc. Since $\boldsymbol{v}_1 \neq 0$, we must have $\alpha_1 = 0$.

Taking dot product of equation 4.1 with $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ etc similarly we get $\alpha_2 = 0$, $\alpha_3 = 0$ etc.

So if we can find an orthogonal set, it is automatically linearly independent.

4.11 Gram-Schmidt Algorithm

Suppose $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$ is any set of linearly independent vectors. We form an orthogonal set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ with the same span.

Stage 1: Let

$$\boldsymbol{v}_1 = \boldsymbol{u}_1.$$

Stage 2: We want a vector orthogonal to $\boldsymbol{u}_1$, so we take the component of $\boldsymbol{u}_2$ orthogonal to $\boldsymbol{u}_1$. By the projection formula (equation 2.1) this is

$$\boldsymbol{v}_2 = \boldsymbol{u}_2 - \frac{\boldsymbol{u}_2 \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1} \boldsymbol{v}_1.$$

Then $\boldsymbol{v}_2$ is orthogonal to $\boldsymbol{v}_1$, by construction. Note that $\frac{\boldsymbol{u}_2 \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}$ is just a scalar. Check of orthogonality:

$$
\begin{aligned}
\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 &= \boldsymbol{v}_1 \cdot \left(\boldsymbol{u}_2 - \frac{\boldsymbol{u}_2 \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1} \boldsymbol{v}_1 \right) \\
&= \boldsymbol{v}_1 \cdot \boldsymbol{u}_2 - \frac{\boldsymbol{u}_2 \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1} (\boldsymbol{v}_1 \cdot \boldsymbol{v}_1) \\
&= \boldsymbol{v}_1 \cdot \boldsymbol{u}_2 - \boldsymbol{u}_2 \cdot \boldsymbol{v}_1 \\
&= 0
\end{aligned}
$$

Since $v_1 = u_1$ and v_2 is given as a linear combination of u_1, v_2 by the formula above, v_1 and v_2 are contained in the 2 dimensional space spanned by $\{u_1, u_2\}$. But orthogonal vectors are linearly independent, so we have found 2 linearly independent vectors in a 2 dimensional space. Thus $\{v_1, v_2\}$ is a basis for span$\{u_1, u_2\}$. So span$\{u_1, u_2\}$ = span$\{v_1, v_2\}$.

At stage 2 we have found a orthogonal set $\{v_1, v_2\}$ of 2 vectors, with the same span as the first two given vectors u_1, u_2.

Stage 3:
$$v_3 = u_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1}v_1 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2}v_2.$$

One can check that then $\{v_1, v_2, v_3\}$ is an orthogonal set of 3 vectors with the same span as $\{u_1, u_2, u_3\}$.

Stage 4:
$$v_4 = u_4 - \frac{u_4 \cdot v_1}{v_1 \cdot v_1}v_1 - \frac{u_4 \cdot v_2}{v_2 \cdot v_2}v_2 - \frac{u_4 \cdot v_3}{v_3 \cdot v_3}v_3.$$

Etc.

The set produced is orthogonal. If we want to make it orthonormal, we can replace each v_i by $v_i/\|v_i\|$ at the end.

4.11.1   Example

Let $u_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix}$, $u_3 = \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}$ in $\mathbb{R}^4$. Find an orthogonal set of vectors with the same span.

$$\boxed{\text{Let } v_1 = u_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix}.}$$

Let

$$\boldsymbol{v}_2 = \boldsymbol{u}_2 - \frac{\boldsymbol{u}_2 \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}\boldsymbol{v}_1$$

$$= \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix} - \frac{3}{6}\begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/2 \\ 2 \\ -1/2 \end{pmatrix}$$

Now let

$$\boldsymbol{v}_3 = \boldsymbol{u}_3 - \frac{\boldsymbol{u}_3 \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}\boldsymbol{v}_1 - \frac{\boldsymbol{u}_3 \cdot \boldsymbol{v}_2}{\boldsymbol{v}_2 \cdot \boldsymbol{v}_2}\boldsymbol{v}_2$$

$$= \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} - \frac{-2}{6}\begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix} - \frac{3}{9/2}\begin{pmatrix} 0 \\ -1/2 \\ 2 \\ -1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 2/3 \\ -4/3 \\ -1/3 \\ 0 \end{pmatrix}$$

So an orthogonal set of vectors with the same span as $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$

An orthonormal basis is $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\}$ where $\boldsymbol{w}_i = \frac{\boldsymbol{v}_i}{\|\boldsymbol{v}_i\|}$, $i = 1, 2, 3$.

Since $\|\boldsymbol{v}_1\| = \sqrt{6}$, $\|\boldsymbol{v}_2\| = \sqrt{18}/2 = 3\sqrt{2}/2$, $\|\boldsymbol{v}_3\| = \frac{\sqrt{21}}{3}$ we get the orthonormal basis

$$\{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\} = \left\{ \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \frac{1}{3\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 4 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{21}}\begin{pmatrix} 2 \\ -4 \\ -1 \\ 0 \end{pmatrix} \right\}.$$

4.11.2 Orthogonal Bases Exist

Every finite-dimensional vector space has an orthogonal basis.

Proof: Pick any basis and apply the Gram-Schmidt process above.

5 Inverses

5.1 The Identity Matrix

An $n \times n$ matrix $(m = n)$ is called a *square matrix* of *order n*. The diagonal containing the entries

$$a_{11}, a_{22}, \cdots, a_{nn}$$

is called the *principal diagonal* (or *main diagonal*) of A. If the entries above this diagonal are all zero then A is called *lower* triangular. If all the entries below the diagonal are zero, A is called *upper triangular*

$$
\begin{pmatrix}
a_{11} & 0 & 0 & \cdots & 0 \\
\vdots & a_{22} & 0 & \cdots & 0 \\
& & \ddots & \ddots & \vdots \\
\vdots & & & \ddots & 0 \\
a_{n1} & \cdots & & \cdots & a_{nn}
\end{pmatrix}
\qquad
\begin{pmatrix}
a_{11} & \cdots & & \cdots & a_{1n} \\
0 & a_{22} & & & \vdots \\
0 & 0 & \ddots & & \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & a_{nn}
\end{pmatrix}
$$
$$\text{lower triangular} \qquad\qquad\qquad \text{upper triangular}$$

If elements above *and* below the principal diagonal are zero, so

$$a_{ij} = 0 \quad, \quad i \neq j$$

then A is called a *diagonal matrix*.

5.1.1 Example

$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ is a diagonal 3×3 matrix.

5.1.2 Definition: Identity matrix

The $n \times n$ *identity matrix* $I = I_n$, is the diagonal matrix whose entries are all 1:

$$
I = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & & & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & 1
\end{pmatrix}.
$$

5.1.3   Comments on the identity matrix

(1) It is easily checked that I acts as the multiplicative identity for $n \times n$ matrices:

$$IA = AI = A$$

Eg. for 2×2 case we have

$$IA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = A = AI.$$

(2) The jth column vector of I is the coordinate vector

$$\boldsymbol{e}_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j.$$

5.2   Definition: Inverse

Let I denote the identity matrix.

A square matrix A is *invertible* (or *non-singular*) if there exists a matrix B such that

$$AB = BA = I.$$

B is called the *inverse* of A and is denoted A^{-1}. A matrix that is not invertible is also said to be *singular*.

5.2.1   Inverse for the 2 × 2 case

Let A be a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Set $B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Then

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} = BA.$$

Let

$$\Delta = ad - bc.$$

If $\Delta \neq 0$, then

$$A^{-1} = \frac{1}{\Delta} B = \frac{1}{\Delta} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

We call Δ the *determinant* of A. We will consider determinants in more detail later (chapter 7). The problem of obtaining inverses of larger square matrices will also be considered later (section 6.4.3).

5.2.2 Example

Find the inverse matrix of $A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$. Check your answer.

$\Delta = 1 \times 5 - 2 \times 3 = -1.$

$$\Rightarrow A^{-1} = \frac{1}{-1} \begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}.$$

Check whether $AA^{-1} = A^{-1}A = I$:
Multiplying the two matrices A and A^{-1} gives

$$\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So A is invertible with inverse

$$\begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}.$$

5.2.3 Example

Show that $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$ is not invertible.

Suppose A has inverse $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Then $\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{cases} a + 2c = 1 & (1) \\ b + 2d = 0 & (2) \\ 3a + 6c = 0 & (3) \\ 3b + 6d = 1 & (4) \end{cases}.$

Hence $3 \cdot (1) - (3) \Rightarrow 0 = 3$. This is impossible, so A can't have an inverse.
Note that $\Delta = 6 - 2 \cdot 3 = 0$ in this case.

5.2.4 Example: Inverse of a 3×3 Matrix

There is no simple formula for the inverse of a 3×3 (or larger) matrix.

Let $A = \begin{pmatrix} 2 & -3 & -1 \\ 1 & -2 & -3 \\ -2 & 2 & -5 \end{pmatrix}$ and $B = \begin{pmatrix} 16 & -17 & 7 \\ 11 & -12 & 5 \\ -2 & 2 & -1 \end{pmatrix}$. Show that B is the inverse of A.

Just multiply the two matrices together.

$$AB = \begin{pmatrix} 2 & -3 & -1 \\ 1 & -2 & -3 \\ -2 & 2 & -5 \end{pmatrix} \begin{pmatrix} 16 & -17 & 7 \\ 11 & -12 & 5 \\ -2 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

And, similarly

$$BA = \begin{pmatrix} 16 & -17 & 7 \\ 11 & -12 & 5 \\ -2 & 2 & -1 \end{pmatrix} \begin{pmatrix} 2 & -3 & -1 \\ 1 & -2 & -3 \\ -2 & 2 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We will learn a systematic way of finding B later (section 6.4.3).

5.3 Invertible Matrices and Linear Independence

As we have seen, not every square matrix has an inverse.

5.3.1 Theorem (existence of inverse)

A square matrix A has an inverse if and only if the columns of A are linearly independent.

Proof:

Suppose A is invertible. We must show its columns are linearly independent. We can isolate the columns with the following technique:

$$Ae_j = \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j$$

$$= \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix} = \begin{array}{l} j\text{th} \quad \text{column} \\ \text{vector of} \\ A. \end{array}$$

Now suppose there is a linear relation among the columns of A. By the previous calculation, we can write this relation as

$$0 = \alpha_1 Ae_1 + \alpha_2 Ae_2 + \cdots + \alpha_n Ae_n$$

Factorize A:

$$\begin{aligned} 0 &= A(\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n) \\ &= A\begin{pmatrix} \alpha_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + A\begin{pmatrix} 0 \\ \alpha_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \ldots + A\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \alpha_n \end{pmatrix} = A\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-1} \\ \alpha_n \end{pmatrix}. \end{aligned}$$

Multiplication on the left by A^{-1} gives

$$0 = \underbrace{A^{-1}A}_{I}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = I\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

$$\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$$

so the column vectors of A are linearly independent.

Conversely, it can be shown that if columns of A are linearly independent then A is invertible.

5.3.2 Example

From example 4.3.3, the columns of

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

are linearly dependent. Therefore A is not invertible.

5.3.3 Example

From example 4.9.1 the columns of

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

are linearly independent, and therefore A has an inverse.

5.3.4 Another Linear Independence Test

Theorem 5.3.1 can also be used in the other direction to give another method of testing if n vectors in $\mathbb{R}^n$ are linearly independent. Put the vectors (as columns) into a matrix A. If A is invertible, then the vectors are linearly independent. If A is not invertible, they are linearly dependent.

Disadvantage of this method: only works for n vectors in $\mathbb{R}^n$ (otherwise the matrix is not square).

5.3.5 Example

Determine whether vectors

$$\boldsymbol{v}_1 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \quad , \quad \boldsymbol{v}_2 = \begin{pmatrix} -3 \\ -2 \\ 2 \end{pmatrix} \quad , \quad \boldsymbol{v}_3 = \begin{pmatrix} -1 \\ -3 \\ -5 \end{pmatrix} \qquad \text{are linearly independent.}$$

Put the vectors into the columns of a matrix: $A = \begin{pmatrix} \boldsymbol{v}_1 | \boldsymbol{v}_2 | \boldsymbol{v}_3 \end{pmatrix}$.

So, let

$$A = \begin{pmatrix} 2 & -3 & -1 \\ 1 & -2 & -3 \\ -2 & 2 & -5 \end{pmatrix}.$$

So $\{v_1, v_2, v_3\}$ are linearly independent iff A is invertible.
We saw in example 5.2.4 that A is invertible, therefore the given vectors are linearly independent.

We can see from this example: swapping columns of a matrix does not effect whether or not the matrix is invertible.

5.3.6 Properties of inverses

(1) A has at most **one** inverse

(2) If A, B are invertible so too is AB and

$$(AB)^{-1} = B^{-1}A^{-1} \quad (not\ A^{-1}B^{-1}!)$$

(3) A is invertible if and only if A^T is invertible

(4) If A is invertible then $(A^T)^{-1} = (A^{-1})^T$.

(5) A is invertible if and only if the columns of A are linearly independent.

(6) A is invertible if and only if the rows of A are linearly independent.

5.3.7 Proof

(1) Suppose B, C are both inverses of A, so

$$AB = BA = I \quad , \quad AC = CA = I.$$

Then $B = BI = B(AC) = (BA)C = IC = C$.

Therefore the inverse, if it exists, is *unique*.

(2) Suppose A, B have inverses A^{-1}, B^{-1} respectively. We have to show that AB has inverse $B^{-1}A^{-1}$. So we just multiply AB by $B^{-1}A^{-1}$, and check that we get the identity matrix I:

$$
\begin{aligned}
(AB)B^{-1}A^{-1} &= A(BB^{-1})A^{-1} \\
&= AIA^{-1} = AA^{-1} = I
\end{aligned}
$$

and similarly

$$(B^{-1}A^{-1})(AB) = I.$$

Therefore AB is invertible with inverse

$$(AB)^{-1} = B^{-1}A^{-1}.$$

(3) and (4) are an exercise:

We prove (4). Assume that A is invertible. We have to show that the inverse of A^T is $(A^{-1})^T$. To show that one matrix is the inverse of the other, we just multiply them together and check we get the identity. We will need the fact that $(AB)^T = B^T A^T$. Setting $B = A^{-1}$ in this, we have

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I.$$

Similarly

$$A^T (A^{-1})^T = (A^{-1}A)^T = I^T = I.$$

Therefore if A is invertible then A^T is invertible with inverse

$$(A^T)^{-1} = (A^{-1})^T$$

This also proves half of (3), namely A invertible implies A^{-1} invertible. To prove the converse, just replace A with A^T in the above calculation. This concludes the proof of both (3) and (4).

(5) is Theorem 5.3.1.

(6) $\qquad A$ is invertible $\overset{3}{\iff} A^T$ is invertible

$\qquad\qquad\qquad\overset{5}{\iff}$ columns of A^T are linearly independent

$\qquad\qquad\qquad\iff$ rows of A are linearly independent.

Notes

6   Gaussian elimination

6.1   Simultaneous equations

One of the main applications of matrices is solving simultaneous equations of the form

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m.
\end{aligned}
$$

Here x_1, x_2, $\cdots\cdots$, x_n are the variables (unknowns) and the coefficients a_{ij} and b_i are given real numbers.

This set of equations can be written using matrix multiplication as

$$
A\boldsymbol{x} = \boldsymbol{b} \tag{6.1}
$$

where $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is a column vector of unknowns, $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$ is a given $m \times 1$ matrix (column vector) and $A = (a_{ij})$ is the $m \times n$ matrix of coefficients.

6.1.1   Special case: n equations in n unknowns.

In the special case that A is $n \times n$ and invertible, we may solve (6.1) by multiplication on the left by A^{-1} to give

$$
\underbrace{A^{-1}A}_{I}\,\boldsymbol{x} = A^{-1}\boldsymbol{b} \Rightarrow \text{ unique solution } \boldsymbol{x} = A^{-1}\boldsymbol{b}.
$$

So if A is invertible, then the system $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution, for each $\boldsymbol{b}$. This is often inaccurately remembered as "n equations in n unknowns have a unique solution". This statement is only true if A is invertible.

The method of multiplying by A^{-1} only works if A is square and A is invertible.

We return to consideration of the general case. An efficient way of solving a system is by *Gaussian elimination*.

6.2 Gaussian elimination

We now illustrate the process of Gaussian elimination. Given a set of equations, the idea is to performs operations, called *elementary row operations*, in order to simplify the equations to the point where the solution may be obtained easily.

Solve the following system of equations:

$$
\begin{aligned}
2x_1 &- 3x_2 &- x_3 &= 1 \\
x_1 &- 2x_2 &- 3x_3 &= 2 \\
-2x_1 &+ 2x_2 &- 5x_3 &= 3
\end{aligned}
$$

Swap the order of the first two equations:

$$
\begin{aligned}
x_1 &- 2x_2 &- 3x_3 &= 2 \\
2x_1 &- 3x_2 &- x_3 &= 1 \\
-2x_1 &+ 2x_2 &- 5x_3 &= 3
\end{aligned}
$$

Subtract 2 times first equation from second:

$$
\begin{aligned}
x_1 &- 2x_2 &- 3x_3 &= 2 \\
 & x_2 &+ 5x_3 &= -3 \\
-2x_1 &+ 2x_2 &- 5x_3 &= 3
\end{aligned}
$$

Add 2 times first equation to third:

$$
\begin{aligned}
x_1 &- 2x_2 &- 3x_3 &= 2 \\
 & x_2 &+ 5x_3 &= -3 \\
 &- 2x_2 &- 11x_3 &= 7
\end{aligned}
$$

Add 2 times second equation to third:

$$
\begin{aligned}
x_1 &- 2x_2 &- 3x_3 &= 2 \\
 & x_2 &+ 5x_3 &= -3 \\
 & &- x_3 &= 1
\end{aligned}
$$

Multiply third equation by -1:

$$
\begin{aligned}
x_1 &- 2x_2 &- 3x_3 &= 2 \\
 & x_2 &+ 5x_3 &= -3 \\
 & & x_3 &= -1
\end{aligned}
$$

Now we can solve by *back substitution:*
Third equation gives $x_3 = -1$.
Substitute x_3 into second: $x_2 - 5 = -3$ so $x_2 = 2$.
Substitute x_2, x_3 into first: $x_1 - 4 + 3 = 2$ so $x_1 = 3$.

This process is called Gaussian elimination. We call this simplified set of equations

$$\begin{aligned} x_1 \quad - \quad 2x_2 \quad - \quad 3x_3 \quad &= \quad 2 \\ x_2 \quad + \quad 5x_3 \quad &= \quad -3 \\ x_3 \quad &= \quad -1 \end{aligned}$$

the *reduced form* of the equations. Once in reduced form the set of equations is easily solved by back substitution.

In this example we repeatedly performed the following operations:

(1) Interchange the order of two equations: swap the ith and j equations.

(2) Multiply an equation through by a non-zero number α.

(3) Add α times the jth equation to the ith equation (where $i \neq j$).

Performing these operations does not change the solution of the equations. But eventually we obtained such a simple system that we could easily find the solution.

6.2.1 Gaussian elimination using matrices

The above procedure may be described most conveniently in terms of matrices. The information present in a set of simultaneous equations

$$\underset{m \times n}{A} \underset{n \times 1}{\boldsymbol{x}} = \underset{m \times 1}{\boldsymbol{b}}$$

may be summarized by constructing a new matrix

$$(A \mid \boldsymbol{b})$$

A with one extra column, $\boldsymbol{b}$. This is called the *augmented matrix*. Instead of manipulating equations we manipulate the corresponding augmented matrix. This way we keep writing to a minimum.

The augmented matrix for the set of equations in the previous example 6.2 may be written

$$\left(\begin{array}{ccc|c} 2 & -3 & -1 & 1 \\ 1 & -2 & -3 & 2 \\ -2 & 2 & -5 & 3 \end{array} \right)$$

Always keep in mind that each row of the augmented matrix represents an equation. The last column is the rhs. The other columns are the coefficients of the variables. Eg the last row above corresponds to $-2x_1 + 2x_2 - 5x_3 = 3$.

Another advantage of the augmented matrix notation is that if we want to solve the equations

$$A\boldsymbol{x} = \boldsymbol{b}, \quad \text{and} \quad A\boldsymbol{y} = \boldsymbol{c},$$

we can do both at once by forming the augmented matrix

$$(A \mid \boldsymbol{b} \mid \boldsymbol{c}).$$

6.2.2 Elementary row operations (EROs)

We rewrite the three allowable operations on equations discussed above, in terms of their effect on the augmented matrix. Call the row vectors of the augmented matrix $r_1, r_2, \ldots$ etc. Since each row corresponds to an equation, the operations are:

(1) Interchange rows i and j, written $r_i \longleftrightarrow r_j$.

(2) Replace r_i with αr_i, $\alpha \in \mathbb{R}$, $\alpha \neq 0$, written $r_i := \alpha r_i$

(3) Replace r_i with $r_i + \alpha r_j$, $\alpha \in \mathbb{R}$, where r_j is another row of A, written $r_i := r_i + \alpha r_j$.

These three operations are known as *elementary row operations* (EROs). In general we say that $m \times n$ matrices A, B are *row equivalent*, written $A \sim B$, if and only if they are related by a series of EROs.

To solve a system of equations:

- Set up the augmented matrix $(A \mid \boldsymbol{b})$ representing the system.

- Apply EROs to $(A \mid \boldsymbol{b})$ to reduce it to a simpler matrix

- Eventually the system is so simple that the solution may be read off.

This is one of the most efficient procedures for solving linear equations.

6.2.3 Example

Redo the previous example 6.2, using row reductions. You should check that we are doing exactly the same calculations as before, just with less writing.

The augmented matrix is:

$$
\left(\begin{array}{ccc|c}
2 & -3 & -1 & 1 \\
1 & -2 & -3 & 2 \\
-2 & 2 & -5 & 3
\end{array} \right)
$$

Swap the first two rows: $r_1 \leftrightarrow r_2$.

$$
\left(\begin{array}{ccc|c}
2 & -3 & -1 & 1 \\
1 & -2 & -3 & 2 \\
-2 & 2 & -5 & 3
\end{array} \right)
\sim
\left(\begin{array}{ccc|c}
1 & -2 & -3 & 2 \\
2 & -3 & -1 & 1 \\
-2 & 2 & -5 & 3
\end{array} \right)
$$

Subtract 2 times row 1 from row 2: $r_2 \to r_2 - 2r_1$.

$$\sim \begin{pmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 5 & -3 \\ -2 & 2 & -5 & 3 \end{pmatrix}$$

Add 2 times row 1 to row 3: $r_3 \to r_3 + 2r_1$.

$$\sim \begin{pmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 5 & -3 \\ 0 & -2 & -11 & 7 \end{pmatrix}$$

Add 2 times row 2 to row 3: $r_3 \to r_3 + 2r_2$.

$$\sim \begin{pmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 5 & -3 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

Multiply row 3 by -1: $r_3 \to -r_3$.

$$\sim \begin{pmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 5 & -3 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

This is called a *Gauss reduced form*, or *row-echelon form*. The solution can now be read off directly by back substitution as before.

r_3 gives $x_3 = -1$.

Substitute x_3 into r_2: $x_2 - 5 = -3$ so $x_2 = 2$.

Substitute x_2, x_3 into r_1: $x_1 - 4 + 3 = 2$ so $x_1 = 3$.

6.2.4 The Gaussian algorithm

To solve

$$\underset{m\times n}{A}\;\underset{n\times 1}{x}\;=\;\underset{m\times 1}{b}\,,$$

first set up the augmented matrix

$$(A \mid b)\,.$$

Then

- Find a row with entry $\alpha \neq 0$ in column 1. (If none go to column 2)

- Multiply row by $\frac{1}{\alpha}$ to make leading entry 1

- Swap rows to bring the 1 to the top

$$\begin{pmatrix} \boxed{1} & \cdot & \cdot & \cdot \\ * & \cdot & \cdot & \cdot \\ * & \cdot & \cdot & \cdot \\ \vdots & & & \end{pmatrix}$$

- Subtract appropriate multiples of row 1 to clear out the first column

$$\begin{pmatrix} \boxed{1} & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ \vdots & & & \end{pmatrix}$$

First column is now done.

Repeat on second column:

Find a row (not the first) with entry $\alpha \neq 0$ in column 2. (If none exists, go on to third column). Make the leading entry 1, and clear out all the entries below this 1:

$$\begin{pmatrix} \boxed{1} & \cdot & \cdot & \cdot \\ 0 & \boxed{1} & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \\ \vdots & & & \end{pmatrix}$$

Then go on to the third column etc.

6.2.5 Gauss reduced form or row echelon form

When solving a system $A\boldsymbol{x} = \boldsymbol{b}$, we can *stop* doing EROs on $(A|\boldsymbol{b})$ once A is "simple enough" to find the solution by back substitution. This means we reduce $(A|\boldsymbol{b})$ until:

- All rows consisting entirely of 0's are at the bottom.

- The first non-zero entry (leading entry) of each row is 1, and appears in a column to the right of the leading entry of the row above it.

- All entries in a column below a leading entry are zeros.

The reduced matrix is called the *Gauss reduced form* (or *row echelon form*) of A. For example:

$$\left(\begin{array}{ccccc|c} \boxed{1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \boxed{1} & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \boxed{1} & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdot \end{array} \right)$$

The leading entries $\boxed{1}$ leading entry (pivot), must be 1 and must have zeros below them. Not every column must have a pivot. Above, the third column has no pivot.

The leading entries must form a "stair case" from top left to bottom right.

(Note that a matrix A may have more than one Gauss reduced form. Eg the entries marked $\cdot$ above are not unique. It doesn't matter which is used.) Tips:

- Try to do row reductions that lead to the easiest algebra.

- Try to avoid fractions.

- Do not neglect row swaps to take advantage of existing 1's in the matrix.

6.2.6 Example

Which of the following matrices are in Gauss reduced form (row echelon form)?

$$A = \left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{array} \right), \quad B = \left(\begin{array}{cccc} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array} \right), \quad C = \left(\begin{array}{cccc} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

A, B are row-reduced. C is not because the leading 1 in the second column is not to the right of the leading 1 in the 1st column.

6.2.7   Inconsistent Equations

It is not always true that m equations in n unknowns must have a solution. Consider the following example.

Solve

$$
\begin{aligned}
x_1 + 2x_2 - x_3 + 2x_4 &= 4 \\
2x_1 + 7x_2 + x_3 + x_4 &= 14 \\
3x_1 + 8x_2 - x_3 + 4x_4 &= 17
\end{aligned}
$$

The augmented matrix is

$$
\begin{pmatrix}
1 & 2 & -1 & 2 & 4 \\
2 & 7 & 1 & 1 & 14 \\
3 & 8 & -1 & 4 & 17
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 2 & -1 & 2 & 4 \\
0 & 3 & 3 & -3 & 6 \\
0 & 2 & 2 & -2 & 5
\end{pmatrix}
\begin{aligned}
&(r_2 \to r_2 - 2r_1) \\
&(r_3 \to r_3 - 3r_1)
\end{aligned}
$$

$$
\sim
\begin{pmatrix}
1 & 2 & -1 & 2 & 4 \\
0 & 1 & 1 & -1 & 2 \\
0 & 2 & 2 & -2 & 5
\end{pmatrix}
\quad (r_2 \to \tfrac{1}{3}r_2)
$$

$$
\sim
\begin{pmatrix}
1 & 2 & -1 & 2 & 4 \\
0 & 1 & 1 & -1 & 2 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\quad (r_3 \to r_3 - 2r_2)
$$

This is in row echelon form. The bottom line corresponds to:

$$
0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 1,
$$

which is impossible. Therefore there are *no solutions* to the system.

6.2.8 How to recognize an inconsistent system

The Gauss reduced form of the augmented matrix will have one or more rows of the form $(0\ 0\ \ldots\ 0\ \mid\ a)$ where $a \neq 0$. This leads to an impossible equation $0x_1 + 0x_2 + \cdots + 0x_n = a \neq 0$.

Geometrically, a linear equation in $\mathbb{R}^2$ represents a line. An equation in $\mathbb{R}^3$ represents a plane etc. Solving 2 equations in $\mathbb{R}^2$ thus corresponds to finding the intersection of 2 lines in $\mathbb{R}^2$. *Usually* two lines in $\mathbb{R}^2$ have a unique point of intersection. But not always: parallel lines do not.

In fact parallel lines do not intersect at all, so there are no solutions—unless the lines are right on top of each other, in which case they intersect in infinitely many points.

Solving 3 equations in $\mathbb{R}^3$ corresponds to finding the intersection of 3 planes. Usually three planes have a unique point of intersection, but not always. Similarly for equations in more variables. This underlies the next two examples.

6.2.9 Example

A system of equations may have more than one solution, i.e. infinitely many solutions.

Solve

$$
\begin{aligned}
x_1 + x_2 &= 0 \\
3x_1 + 3x_2 + x_3 &= 0.
\end{aligned}
$$

The augmented matrix is

$$
\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right)
$$

Thus our reduced system is

$$
\begin{aligned}
x_1 + x_2 &= 0 \\
x_3 &= 0
\end{aligned}
$$

Take $x_2 = \alpha$. Then

$$
x_1 = -x_2 = -\alpha, \qquad x_2 = \alpha, \qquad x_3 = 0
$$

So the solution is: $\boldsymbol{x} = \alpha \left(\begin{array}{c} -1 \\ 1 \\ 0 \end{array} \right)$, for any $\alpha \in \mathbb{R}$.

6.2.10   Example

Even if we have as many equations as unknowns, we may still have more than one solution, i.e. infinitely many solutions, or no solutions.

Solve the system of equations

$$
\begin{aligned}
x_1 + 3x_2 - 2x_3 &= 2 \\
-x_1 + x_2 - 2x_3 &= 2 \\
2x_1 + 4x_2 - 2x_3 &= 2
\end{aligned}
$$

The augmented matrix is:

$$
\begin{pmatrix}
1 & 3 & -2 & 2 \\
-1 & 1 & -2 & 2 \\
2 & 4 & -2 & 2
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 3 & -2 & 2 \\
0 & 4 & -4 & 4 \\
0 & -2 & 2 & -2
\end{pmatrix}
\quad
\begin{aligned}
(r_2 &\to r_2 + r_1) \\
(r_3 &\to r_3 - 2r_1)
\end{aligned}
$$

$$
\sim (r_2 \to \frac{r_2}{4})
\begin{pmatrix}
1 & 3 & -2 & 2 \\
0 & 1 & -1 & 1 \\
0 & -2 & 2 & -2
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 3 & -2 & 2 \\
0 & 1 & -1 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\quad (r_3 \to r_3 + 2r_2)
$$

(Gauss reduced form).
The last row is equivalent to

$$
0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 0, \quad \text{which gives no information.}
$$

Therefore we effectively have just 2 equations in 3 unknowns

$$
\begin{aligned}
x_1 + 3x_2 - 2x_3 &= 2 \\
x_2 - x_3 &= 1
\end{aligned}
$$

No leading 1 in the third column (non-pivotal). So set $x_3 = \alpha$ arbitrary, and back substitute into r_2 and r_1 to give

$$
\begin{aligned}
x_2 &= 1 + x_3 = 1 + \alpha, \\
x_1 &= 2 - 3x_2 + 2x_3 = -1 - \alpha
\end{aligned}
$$

This means, for *any* choice of $\alpha \in \mathbb{R}$, a solution is:

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 - \alpha \\ 1 + \alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

The last equation is the equation of a line in $\mathbb{R}^3$: $\boldsymbol{x}$ is given by a position vector $\boldsymbol{p} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ plus α times a vector $\boldsymbol{v} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ indicating direction.

Thus geometrically the three planes intersect along a common line rather than at a single point. Every solution $\boldsymbol{x}$ corresponds to a point on this line obtained by a particular choice of α.

6.2.11 How to recognize non-unique solutions

The system is consistent, and the Gauss reduced matrix will have one or more columns without a pivot (leading entry).

For example, if the second column does not have a leading 1 then there will never be an equation we can use to isolate x_2, so x_2 will be a free variable: we will be able to assign it any value (α say), and still solve the system. In a big system we may have several free variables.

6.3 The general solution

6.3.1 General form of solutions

In general a system of equations

$$\underset{m\times n}{A}\ \underset{n\times 1}{\boldsymbol{x}} = \underset{m\times 1}{\boldsymbol{b}}$$

will have one of the following

- A unique solution

- An infinite number of solutions

- No solution.

Which behaviour occurs can be seen from the Gauss reduced form of the matrix.

We shall show that if there are an infinite number of solutions, they will all be of the form $\boldsymbol{p} + \boldsymbol{y}$ where $\boldsymbol{p}$ is a particular solution and $\boldsymbol{y}$ is the part arising from the free variables. We shall show that $\boldsymbol{y}$ is a solution of $A\boldsymbol{x} = \boldsymbol{0}$.

It is useful to introduce some notation.

6.3.2 The Nullspace of a matrix

Let A be a matrix. The set of vectors $\boldsymbol{x}$ with $A\boldsymbol{x} = \boldsymbol{0}$ is called the *nullspace* of A, denoted $\mathcal{N}(A)$.

Theorem: $\mathcal{N}(A)$ is a vector space.

Proof:

(o) $A\boldsymbol{0} = \boldsymbol{0}\quad \Rightarrow \boldsymbol{0} \in \mathcal{N}(A)\quad \Rightarrow \mathcal{N}(A)$ is non-empty.

(i) $\boldsymbol{v}, \boldsymbol{w} \in \mathcal{N}(A)\quad \Rightarrow A\boldsymbol{v} = \boldsymbol{0}, A\boldsymbol{w} = \boldsymbol{0}$
$\Rightarrow A(\boldsymbol{v} + \boldsymbol{w}) = A\boldsymbol{v} + A\boldsymbol{w} = \boldsymbol{0} + \boldsymbol{0} = \boldsymbol{0}\quad \Rightarrow \boldsymbol{v} + \boldsymbol{w} \in \mathcal{N}(A)$

(ii) If $\boldsymbol{v} \in \mathcal{N}(A)$, $\alpha \in \mathbb{R}$ then $A\boldsymbol{v} = \boldsymbol{0}$
$\Rightarrow A(\alpha\boldsymbol{v}) = \alpha A\boldsymbol{v} = \alpha\boldsymbol{0} = \boldsymbol{0}\quad \Rightarrow \alpha\boldsymbol{v} \in \mathcal{N}(A)$

Hence $\mathcal{N}(A)$ is a vector space.

Notice that this means either $\mathcal{N}(A) = \{\boldsymbol{0}\}$ or else it is infinite. If there is any non-zero solution $\boldsymbol{v}$ of $A\boldsymbol{v} = \boldsymbol{0}$, there will be infinitely many solutions.

The linear system $A\boldsymbol{x} = \boldsymbol{0}$ is called a *homogeneous system.* (Homogeneous means "all the same".) Thus the nullspace of A is the set of solutions of the homogeneous system.

A linear system $A\boldsymbol{x} = \boldsymbol{b}$ with $\boldsymbol{b} \neq \boldsymbol{0}$ is called *inhomogeneous.*

6.3.3 Relation between homogeneous & inhomogeneous systems

The solution of the homogeneous $A\boldsymbol{x} = \boldsymbol{0}$ and inhomogeneous $A\boldsymbol{x} = \boldsymbol{b}$ systems are related.

Let A be a matrix, and consider the system

$$(*) \quad A\boldsymbol{x} = \boldsymbol{b}.$$

Let $\boldsymbol{p}$ be a fixed solution of (*). Then the complete set of solutions is

$$\{\boldsymbol{x} = \boldsymbol{p} + \boldsymbol{y} \mid \boldsymbol{y} \in \mathcal{N}(A)\}.$$

Proof: If $\boldsymbol{p}$ is a solution and $\boldsymbol{y} \in \mathcal{N}(A)$ then $A(\boldsymbol{p} + \boldsymbol{y}) = A\boldsymbol{p} + A\boldsymbol{y} = \boldsymbol{b} + \boldsymbol{0} = \boldsymbol{b}$. So every $\boldsymbol{x}$ of the form $\boldsymbol{p} + \boldsymbol{y}$ is a solution.

Conversely, if $\boldsymbol{x}$ is a solution then $A\boldsymbol{x} = \boldsymbol{b} = A\boldsymbol{p}$. Thus $A(\boldsymbol{x} - \boldsymbol{p}) = A\boldsymbol{x} - A\boldsymbol{p} = \boldsymbol{b} - \boldsymbol{b} = \boldsymbol{0}$. So let $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{p} \in \mathcal{N}(A)$. Then $\boldsymbol{x} = \boldsymbol{p} + \boldsymbol{y}$ has the stated form.

6.3.4 Example

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 2 & 1 & 0 \end{pmatrix}$ and let $\boldsymbol{b} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$. Calculate $\mathcal{N}(A)$. Also solve the system $A\boldsymbol{x} = \boldsymbol{b}$.

We form the augmented matrix $(A \mid \boldsymbol{0} \mid \boldsymbol{b})$:

$$\left(\begin{array}{ccc|c|c} 1 & 2 & 3 & 0 & 2 \\ 2 & 5 & 8 & 0 & 5 \\ 2 & 1 & 0 & 0 & 1 \end{array} \right) \sim \left(\begin{array}{ccc|c|c} 1 & 2 & 3 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & -3 & -6 & 0 & -3 \end{array} \right) \sim \left(\begin{array}{ccc|c|c} 1 & 2 & 3 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

This is in row echelon form and system is consistent. Since there is no pivot in third column, there will be infinitely many solutions.

Homogeneous case: $x_2 + 2x_3 = 0$, so $x_2 = -2x_3$. Then $x_1 - 4x_3 + 3x_3 = 0$, so $x_1 = x_3$. Letting $x_3 = \alpha$,

$$\mathcal{N}(A) = \left\{ \alpha \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \mid \alpha \in \mathbb{R} \right\}.$$

Inhomogeneous case: $x_3 = \alpha$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$x_2 + 2\alpha = 1 \quad \Rightarrow \quad x_2 = 1 - 2\alpha$$
$$x_1 + 2x_2 + 3\alpha = 2 \Rightarrow x_1 + 2(1 - 2\alpha) + 3x_2 = 2$$
$$\Rightarrow \quad x_1 - \alpha = 0$$
$$\Rightarrow \quad x_1 = \alpha.$$

Then

$$\boldsymbol{x} = \left(\begin{array}{c} \alpha \\ 1 - 2\alpha \\ \alpha \end{array} \right) = \left(\begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) + \alpha \left(\begin{array}{c} 1 \\ -2 \\ 1 \end{array} \right) = \boldsymbol{p} + \boldsymbol{y},$$

where $\boldsymbol{p} = \left(\begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right)$ is a particular solution of $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{y} \in \mathcal{N}(A)$.

6.3.5 Summary

There are 3 possible behaviours for a system:

1. The system $A\boldsymbol{x} = \boldsymbol{b}$ may have no solutions. (Inconsistent case.)

2. If a solution to $A\boldsymbol{x} = \boldsymbol{b}$ exists and $\mathcal{N}(A) = \{\boldsymbol{0}\}$ then $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution.

3. If $A\boldsymbol{x} = \boldsymbol{b}$ has a solution $\boldsymbol{x} = \boldsymbol{p}$ and $\mathcal{N}(A) \neq \{\boldsymbol{0}\}$ then there are infinitely many solutions: all $\boldsymbol{x}$ of the form $\boldsymbol{x} = \boldsymbol{p} + \boldsymbol{y}$ with $\boldsymbol{y} \in \mathcal{N}(A)$.

Important: Note that a homogeneous system is always consistent, because the 0-vector is always a solution.

6.3.6 Invertibility and the Nullspace

Now suppose A is a square matrix. We add some further conditions equivalent to A being invertible:

Theorem: The following are equivalent for a square matrix A:

1. A is invertible.

2. For every $\boldsymbol{b}$, the system $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution

3. $\mathcal{N}(A) = \{\boldsymbol{0}\}$

Proof: $(1) \Rightarrow (2)$ Done earlier: in section 6.1.1.

$(2) \Rightarrow (3)$ If (2) is true then $A\boldsymbol{x} = \boldsymbol{0}$ has a unique solution, which must be $\boldsymbol{x} = \boldsymbol{0}$. So $\mathcal{N}(A) = \{0\}$.

$(3) \Rightarrow (1)$ We use the same trick to isolate the columns of A that we saw in Theorem 5.3.1. $A\boldsymbol{x} = \boldsymbol{0}$ can be expressed as

$$A\boldsymbol{x} = A(x_1\boldsymbol{e}_1 + \cdots \cdots + x_n\boldsymbol{e}_n) = \boldsymbol{0}$$

or

$$x_1 A\boldsymbol{e}_1 + x_2 A\boldsymbol{e}_2 + \cdots \cdots + x_n A\boldsymbol{e}_n = \boldsymbol{0}$$

where $A\boldsymbol{e}_j = j$th column vector of A. Therefore $\boldsymbol{x} = \boldsymbol{0}$ unique solution $\Longleftrightarrow$ column vectors of A linearly independent $\Leftrightarrow$ A is invertible by Theorem 5.3.1.

6.4 Gauss-Jordan elimination

Gauss-Jordan (or simply Jordan) reduction involves performing more EROs on the Gauss reduced form of a matrix until it has the additional property that each leading 1 is the *only* non-zero entry in its column.

Solve
$$
\begin{array}{rcrcrcl}
2x_1 & - & 3x_2 & - & x_3 & = & 1 \\
x_1 & - & 2x_2 & - & 3x_3 & = & 2 \\
-2x_1 & + & 2x_2 & - & 5x_3 & = & 3
\end{array}
$$

6.4.1 Solution

This is example 6.2.3 again. The augmented matrix is:

$$
\left(
\begin{array}{ccc|c}
2 & -3 & -1 & 1 \\
1 & -2 & -3 & 2 \\
-2 & 2 & -5 & 3
\end{array}
\right)
\sim
\left(
\begin{array}{ccc|c}
1 & -2 & -3 & 2 \\
0 & 1 & 5 & -3 \\
0 & 0 & 1 & -1
\end{array}
\right)
$$

This is the Gauss reduced form. But we can continue doing row operations.

$$\sim \begin{pmatrix} 1 & 0 & 7 & -4 \\ 0 & 1 & 5 & -3 \\ 0 & 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$r_1 \to r_1 + 2r_2 \qquad \begin{aligned} r_1 &\to r_1 - 7r_3 \\ r_2 &\to r_2 - 5r_3 \end{aligned}$$

This is the Jordan reduced form. We can read off the solution directly: $x_1 = 3$, $x_2 = 2$, $x_3 = -1$.

6.4.2 The difference between Gauss and Gauss-Jordan reduction

Gauss reduction leads to a matrix with 0's below the leading 1's, such as

$$\begin{pmatrix} \boxed{1} & * & * & * \\ 0 & \boxed{1} & * & * \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

while Gauss-Jordan reduction results in a matrix with 0's above and below the leading 1's such as

$$\begin{pmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Jordan form has the advantage that the solution can be read off directly without any back substitution. Nonetheless, overall Jordan reduction takes more steps than [Gaussian reduction plus back substitution]. Thus Gaussian reduction is usually preferred for solving systems. However Gauss-Jordan reduction is better if we want to solve many systems $A\boldsymbol{x} = \boldsymbol{b}$, $A\boldsymbol{x} = \boldsymbol{c}$ etc with the same coefficient matrix A, since it eliminates the need to do multiple back substitution steps. This is useful in the next section.

6.4.3 Inverses using Gauss–Jordan reduction

Gauss–Jordan reduction is useful for obtaining inverses of matrices.

Let A be an invertible $n \times n$ matrix. We want to find the inverse X of A, so

$$AX = I$$

Now let x_j be the jth column of the matrix X, and recall that e_j is the jth column of I. Breaking the previous matrix equation into one equation for each column gives rise to n equations

$$Ax_j = e_j \qquad (6.2)$$

We can regard each of these as a system to be solved. As we noted when introducing the augmented matrix, we can solve more than one system of equation at once by forming a larger augmented matrix. Eg

$$Ax = b, \ Ax = c \ \Rightarrow \ (A \mid b \ c)$$

Thus we may solve (6.2) by applying Gauss–Jordan elimination to the augmented matrix

$$(A \mid e_1 \ e_2 \ \ldots, e_n) = (A \mid I).$$

Reduce to Gauss-Jordan form

$$(I \mid X)$$

The solution X can now be read off, so $X = A^{-1}$ is the inverse of A. If this reduction cannot be completed, A has no inverse.

Note: actually this only proves $AX = I$. One can show that $XA = I$ also, so that X really is the inverse of A.

6.4.4 Example

Let $A = \begin{pmatrix} 2 & -3 & -1 \\ 1 & -2 & -3 \\ -2 & 2 & -5 \end{pmatrix}$. Calculate A^{-1}. (This is example 5.2.4).

Perform Gauss-Jordan reduction on the augmented matrix $(A \mid I)$:

$$\left(\begin{array}{ccc|ccc} 2 & -3 & -1 & 1 & 0 & 0 \\ 1 & -2 & -3 & 0 & 1 & 0 \\ -2 & 2 & -5 & 0 & 0 & 1 \end{array} \right) \sim \left(\begin{array}{ccc|ccc} 1 & -2 & -3 & 0 & 1 & 0 \\ 2 & -3 & -1 & 1 & 0 & 0 \\ -2 & 2 & -5 & 0 & 0 & 1 \end{array} \right)$$
$$r_1 \longleftrightarrow r_2$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -3 & 0 & 1 & 0 \\ 0 & 1 & 5 & 1 & -2 & 0 \\ 0 & -2 & -11 & 0 & 2 & 1 \end{array} \right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 7 & 2 & -3 & 0 \\ 0 & 1 & 5 & 1 & -2 & 0 \\ 0 & 0 & -1 & 2 & -2 & 1 \end{array} \right)$$
$$\begin{array}{cc} r_2 \to r_2 - 2r_1 & \qquad r_1 \to r_1 + 2r_2 \\ r_3 \to r_3 + 2r_1 & \qquad r_3 \to r_3 + 2r_2 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix} \left.\begin{matrix} 2 & -3 & 0 \\ 1 & -2 & 0 \\ -2 & 2 & -1 \end{matrix}\right) \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left.\begin{matrix} 16 & -17 & 7 \\ 11 & -12 & 5 \\ -2 & 2 & -1 \end{matrix}\right)$$

$$r_3 \to -r_3 \qquad\qquad\qquad r_1 \to r_1 - 7r_3$$
$$r_2 \to r_2 - 5r_3$$

So A is invertible, and $A^{-1} = \begin{pmatrix} 16 & -17 & 7 \\ 11 & -12 & 5 \\ -2 & 2 & -1 \end{pmatrix}$.

Compare this to 5.2.4.

Notes

7 Determinants

7.1 Definition: Determinant

Associated to each square matrix A is a number called the *determinant* of A and denoted $|A|$ or $\det(A)$. It is defined as follows.

- If $A = (a_{ij})$ is 2×2 then

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

  We already encountered this in 5.2.1.

- In general if $A = (a_{ij})$ is $n \times n$, first set

$$C_{ij} = (-1)^{i+j} \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1\ j-1} & a_{1\ j+1} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2\ j-1} & a_{2\ j+1} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1\ 1} & a_{i-1\ 2} & \dots & a_{i-1\ j-1} & a_{i-1\ j+1} & \dots & a_{i-1\ n} \\ a_{i+1\ 1} & a_{i+1\ 2} & \dots & a_{i+1\ j-1} & a_{i+1\ j+1} & \dots & a_{i+1\ n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{n\ j-1} & a_{n\ j+1} & \dots & a_{nn} \end{vmatrix}$$

  called the cofactor of a_{ij}. The $(n-1) \times (n-1)$ determinant is obtained by *omitting* the ith row and jth column from A (indicated by the horizontal and vertical lines in the matrix).

  We then define

$$|A| = a_{11}C_{11} + a_{12}C_{12} + \cdots\cdots + a_{1n}C_{1n}$$

  which gives a *recursive* definition of the determinant.

Observe that $(-1)^{i+j}$ gives the pattern $\begin{pmatrix} + & - & + & - & \cdot & \cdot & \cdot & \cdot \\ - & + & - & + & \cdot & \cdot & \cdot & \cdot \\ + & - & + & - & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & & & & & \\ \cdot\cdot & & & & & & & \end{pmatrix}$

Thus for a 3×3 matrix A we have

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

where

$$C_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad C_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \quad C_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

7.1.1 Example

Find the cofactors C_{13} and C_{23} of the matrix

$$\begin{pmatrix} 3 & 5 & 7 \\ 1 & 0 & 2 \\ 0 & 3 & 0 \end{pmatrix}.$$

$$C_{13} = (-1)^{1+3} M_{13} = 1 \cdot \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} = 3,$$

$$C_{23} = (-1)^{2+3} M_{23} = -1 \cdot \begin{vmatrix} 3 & 5 \\ 0 & 3 \end{vmatrix} = -9.$$

7.1.2 Example

Calculate the determinant $\begin{vmatrix} 3 & 5 & 7 \\ 1 & 0 & 2 \\ 0 & 3 & 0 \end{vmatrix}.$

$$\begin{aligned} \begin{vmatrix} 3 & 5 & 7 \\ 1 & 0 & 2 \\ 0 & 3 & 0 \end{vmatrix} &= (1) \cdot (3) \cdot \begin{vmatrix} 0 & 2 \\ 3 & 0 \end{vmatrix} + (-1) \cdot (5) \cdot \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} + (1) \cdot (7) \cdot \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} \\ &= 3(0 - 6) - 5(0 - 0) + 7(3 - 0) \\ &= 3. \end{aligned}$$

7.2 Properties of Determinants

Property (1): $|A| = |A^T|$

Consider for example the 2×2 case. Then

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \Rightarrow \quad A^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$$

Thus $|A| = a_{11}a_{22} - a_{12}a_{21} = |A^T|$. This holds in general.

This means that any results about the *rows* in a general determinant is also true about the *columns* (since the rows of A^T are the columns of A).

Property (2): The determinant may by taking cofactors along any row (not just the first) or down any column. Eg. for a 3×3 matrix A

$$\begin{aligned} |A| &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \quad \text{(definition, expansion along 1st row)} \\ &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \quad \text{(expansion along 2nd row)} \\ &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} \quad \text{(expansion down 3rd column) etc.} \end{aligned}$$

This is useful if one row or column contains a larger number of zeros.

7.2.1 Examples

(i) Calculate $\begin{vmatrix} 3 & 5 & 7 \\ 1 & 0 & 2 \\ 0 & 3 & 0 \end{vmatrix}$ by (a) expanding down the third column, (b) expanding along the third row. Which is more efficient?

$$\begin{aligned} \begin{vmatrix} 3 & 5 & 7 \\ 1 & 0 & 2 \\ 0 & 3 & 0 \end{vmatrix} &= 7 \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} - 2 \begin{vmatrix} 3 & 5 \\ 0 & 3 \end{vmatrix} = 21 - 18 = 3 \quad \begin{pmatrix} \text{exp. down} \\ \text{3rd. col.} \end{pmatrix} \\ &= -3 \begin{vmatrix} 3 & 7 \\ 1 & 2 \end{vmatrix} = -3(6 - 7) = 3 \quad \begin{pmatrix} \text{exp. along} \\ r_3 \end{pmatrix} \end{aligned}$$

(ii) Find
$$\begin{vmatrix} 3 & 6 & 9 \\ 0 & 0 & 0 \\ 2 & 4 & 5 \end{vmatrix}.$$

$$\begin{vmatrix} 3 & 6 & 9 \\ 0 & 0 & 0 \\ 2 & 4 & 5 \end{vmatrix} = 0 \cdot C_{21} + 0 \cdot C_{22} + 0 \cdot C_{23}$$
$$= 0 \qquad \text{(expanding along } r_2)$$

In general any determinant with a row or column of zeros vanishes.

7.2.2 Effect of Elementary Row Operations on Determinants

The calculation of large determinants from the definition is very tedious. It is much easier to calculate the determinant after row reducing a matrix, because the reduced matrix will contain many zeros. However, performing elementary row operations changes the value of a determinant. Luckily the changes are simple:

Property (3): From (2), a common factor in *all* entries of a row (or column) can be taken out as a factor of the determinant.

7.2.3 Example

$$\begin{vmatrix} 3 & 6 & 9 \\ 0 & 1 & 2 \\ 3 & 4 & 5 \end{vmatrix} = 3C_{11} + 6C_{12} + 9C_{13} \qquad \text{(expanding along } r_1)$$
$$= 3[C_{11} + 2C_{12} + 3C_{13}]$$
$$= 3 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 4 & 5 \end{vmatrix} \qquad \text{(check!)}$$

Property (4): If 2 rows (or columns) of A are interchanged ($r_i \longleftrightarrow r_j$ or $c_i \longleftrightarrow c_j$), $|A|$ changes sign.

7.2.4 Example

We have seen $\begin{vmatrix} 3 & 5 & 7 \\ 1 & 0 & 2 \\ 0 & 3 & 0 \end{vmatrix} = 3$. Interchanging columns 2 and 3 gives

$$\begin{vmatrix} 3 & 7 & 5 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 3 \begin{vmatrix} 3 & 7 \\ 1 & 2 \end{vmatrix} \text{ (expanding along } r_3)$$

$$= -3.$$

Let A be a matrix, and let B be the matrix obtained by interchanging two rows or columns. Then $\det(A) = -\det(B)$. If A has two equal rows (or columns) and these are interchanged then $A = B$ so $\det(A) = -\det(B) = -\det(A)$ so $\det(A) = 0$. Hence:

If A has two equal rows or columns then $\det(A) = 0$.

Property (5): If a multiple of one row (or column) is added to another ($r_i \to r_i + \alpha r_j$ or $c_i \to c_i + \alpha c_j$), the determinant is unchanged.

The above 5 properties give a shortcut for evaluating determinants. We can use a sequence of elementary row operations (and/or column operations) to produce a matrix with a row (or column) with at most one non-zero entry. We then expand along this row (or column) to obtain a smaller determinant, and repeat.

Important: row (and column) operations can change the value of the determinant, so we must always keep track of which operations we perform. (Unlike doing Gaussian elimination.)

7.2.5 Example

Calculate the determinant $\begin{vmatrix} 2 & 5 & 6 & 1 \\ 1 & 2 & 3 & -2 \\ 3 & 2 & 4 & 4 \\ 0 & 2 & 4 & 6 \end{vmatrix}$.

$$\begin{vmatrix} 2 & 5 & 6 & 1 \\ 1 & 2 & 3 & -2 \\ 3 & 2 & 4 & 4 \\ 0 & 2 & 4 & 6 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 5 \\ 1 & 2 & 3 & -2 \\ 0 & -4 & -5 & 10 \\ 0 & 2 & 4 & 6 \end{vmatrix} \quad \begin{matrix} (r_1 \to r_1 - 2r_2) \\ \\ (r_3 \to r_3 - 3r_2) \end{matrix}$$

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$$= (-1) \cdot \begin{vmatrix} 1 & 0 & 5 \\ -4 & -5 & 10 \\ 2 & 4 & 6 \end{vmatrix} \quad \text{(expanding down column 1)}$$

$$= (-1) \cdot \begin{vmatrix} 1 & 0 & 0 \\ -4 & -5 & 30 \\ 2 & 4 & -4 \end{vmatrix} \quad (c_3 \to c_3 - 5c_1)$$

$$= - \begin{vmatrix} -5 & 30 \\ 4 & -4 \end{vmatrix} \quad \text{(expanding along } r_1)$$

$$= (-5) \cdot 4 \cdot \begin{vmatrix} -1 & 6 \\ 1 & -1 \end{vmatrix} = -20(1-6) = 100.$$

Using the definition with cofactors, to calculate an $n \times n$ determinant, we have to find n cofactors, each a $(n-1) \times (n-1)$ determinant. For *each* of these we need to find $(n-1)$ cofactors of size $(n-2) \times (n-2)$ etc. The total number of calculations is about

$$n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 = n!$$

For a 25×25, this is about 10^{25} operations. This would take thousands of years, even on a computer. Doing row reductions, only a few thousand calculations are required, which could be done in a fraction of a second on a computer.

7.2.6 Example

Evaluate $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$. This matrix is called Vandermonde's matrix, and it occurs in applications such as signal processing, error-correcting codes, and polynomial interpolation.

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{vmatrix} \begin{matrix} (r_2 \to r_2 - ar_1) \\ (r_3 \to r_3 - a^2 r_1) \end{matrix}$$

$$= \begin{vmatrix} b-a & c-a \\ b^2-a^2 & c^2-a^2 \end{vmatrix} \quad \text{(expand down first column)}$$

Now we extract a common factor from each of columns 1 and 2 giving

$$= (b-a)(c-a)\begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix}$$

$$= (b-a)(c-a)[c+a-(b+a)]$$

$$= (b-a)(c-a)(c-b).$$

7.2.7 Determinants of Triangular Matrices

Suppose $A = \begin{pmatrix} a_{11} & 0 & \cdots & \cdots & 0 \\ a_{21} & a_{22} & 0 & & \vdots \\ \vdots & & \ddots & & \vdots \\ & & & & 0 \\ a_{n1} & \cdots & \cdots & \cdots & a_{nn} \end{pmatrix}$

is lower triangular. Then by repeated expansion along r_1 we obtain

$$|A| = a_{11} \begin{vmatrix} a_{22} & & \\ \vdots & \ddots & \\ \vdots & & \vdots \\ a_{n2} & \cdots & \cdots & a_{nn} \end{vmatrix} = \ldots = a_{11} \, a_{22} \cdot \ldots \cdot a_{nn},$$

which is the product of the diagonal entries. The same result holds for upper triangular and diagonal matrices.

In particular

$$|I| = 1.$$

7.3 Connection with inverses and systems of linear equations

An important property of determinants is the following.

7.3.1 Fact (product of determinants)

Let A, B be $n \times n$ matrices. Then

$$|AB| = |A| \cdot |B|.$$

7.3.2 Theorem: Invertible matrices

$$A \text{ is invertible} \iff |A| \neq 0.$$

7.3.3 Proof

$\implies$ If A is invertible, then

$$I = AA^{-1}$$
$$\implies \quad 1 = |I| = |AA^{-1}| = |A| |A^{-1}|,$$
$$\text{So } |A| \neq 0.$$

$\impliedby$ Follows from 7.3.4.

Remark: It follows immediately from the proof that if A is invertible, then

$$|A^{-1}| = \frac{1}{|A|},$$

i.e., *the inverse of the determinant is the determinant of the inverse.*

7.3.4 A Theoretical Formula

Actually there is an explicit formula for the inverse of A. If C_{ij} is the cofactor of the i, j entry of A, let C be the matrix of cofactors $C = (C_{ij})$. Then

$$A^{-1} = \frac{1}{|A|} C^T.$$

Here C^T is called the adjoint matrix of A. This formula is useful for some theoretical problems, but is useless as a tool for calculations because it is so time consuming to calculate all the cofactors.

Never use this formula for calculating inverses!

7.3.5 Example

In the 2×2 case we have

$$|A| = \left| \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right| = a_{11}a_{22} - a_{12}a_{21}$$

and co-factors

$$C_{11} = a_{22} \quad , \quad C_{12} = -a_{21} \quad , \quad C_{21} = -a_{12} \quad , \quad C_{22} = a_{11}$$

$$\therefore \quad |A| \neq 0 \Rightarrow A^{-1} = \frac{1}{|A|} \left(\begin{array}{cc} C_{11} & C_{21} \\ C_{12} & C_{22} \end{array} \right) = \frac{1}{|A|} \left(\begin{array}{cc} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{array} \right)$$

in agreement with a previous result, 5.2.1.

7.3.6 Summary of conditions equivalent to invertibility

Let A be an $n \times n$ matrix. Give a summary of all the conditions we have found that are equivalent to A being invertible.

1. A is invertible.

2. There exists an $n \times n$ matrix X with $AX = XA = I$ [Definition of invertible, § 5.2].

3. A^T is invertible   [§ 5.3.6(3)].

4. The columns of A are linearly independent   [Theorem 5.3.1 and § 5.3.6(5)].

5. The rows of A are linearly independent   [§ 5.3.6(6)].

6. $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution $\boldsymbol{x}$ for any $\boldsymbol{b} \in \mathbb{R}$ [§ 6.3.6].

7. $\mathcal{N}(A) = \{\boldsymbol{0}\}$ [§ 6.3.6].

8. The row reduced form of A is I [§ 6.4].

9. $\det(A) \neq 0$ [Theorem 7.3.2].

8 Vector Products in 3-Space

8.1 Definition: Cross product

For vectors v, w in 3-*space* $\mathbb{R}^3$, the *cross* (or *vector*) product of v and w is a vector $v \times w$ such that

(i) $\|v \times w\| = \|v\| \cdot \|w\| \sin \theta$, where θ is the angle between v and w

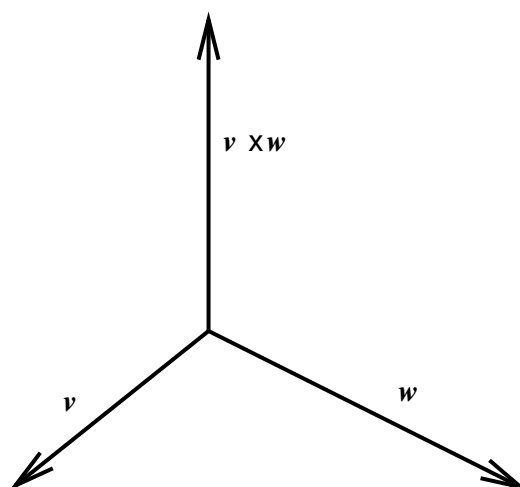(ii) $v \times w$ is perpendicular to v and w, in right hand direction from $v \times w$ (right hand rule).

Figure 11: The vector $v \times w$ is perpendicular to both v and w in $\mathbb{R}^3$

8.1.1 Notes on the cross product

(1) For v, $w \neq \mathbf{0}$

$v \times w = \mathbf{0} \Leftrightarrow \sin \theta = 0 \Leftrightarrow \theta = 0$ or $180°$

$\therefore$ $v \times w = \mathbf{0} \Leftrightarrow v$ and w are parallel (or anti-parallel).

(2) $v \times w = -w \times v$ since they have the same magnitude but opposite direction

(3) $v \times v = \mathbf{0}$, since $\sin 0 = 0$.

(4) If $v \times w \neq \mathbf{0}$, then $v \times w$ is perpendicular to the plane determined by v and w. This is useful for obtaining the equation of a plane in $\mathbb{R}^3$.

8.1.2 Example

$$
\begin{aligned}
i \times i &= j \times j = k \times k = 0 \\
i \times j &= k = -j \times i \quad , \quad j \times k = i = -k \times j \\
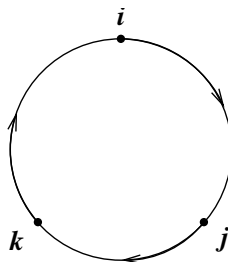k \times i &= j = -i \times k
\end{aligned}
$$

Figure 12: This diagram is an aid in remembering the cross product between unit vectors.

8.1.3 Application: angular momentum

Vector products also arise naturally in physics and engineering eg. a particle of unit charge moving with velocity v in a magnetic field B experiences a force $F = v \times B$. The *angular momentum* of a particle of mass m moving with velocity v is given by $L = m(r \times v)$ where r is the position vector of the particle. This is important for central force problems.

8.1.4 Properties

We have seen that
$$
v \times w = -w \times v
$$
so the commutative law fails. Also we do not have the associative law: ie. in general
$$
v \times (w \times u) \neq (v \times w) \times u
$$
For example, $(i \times i) \times k = 0 \times k = 0$, but $i \times (i \times k) = i \times (-j) = -k$.

However, it can be shown that
$$
\begin{aligned}
v \times (w + u) &= v \times w + v \times u \\
(v + w) \times u &= v \times u + w \times u \\
v \times (\alpha w) &= (\alpha v) \times w = \alpha(v \times w) \quad , \quad \alpha \in \mathbb{R}.
\end{aligned}
$$

Using the above results we have the following theorem in terms of components.

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8.1.5 Theorem: Calculating the cross product

Suppose $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\quad \boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$. Then

$$\boldsymbol{v} \times \boldsymbol{w} \;=\; (v_2 w_3 - v_3 w_2)\boldsymbol{i} + (v_3 w_1 - v_1 w_3)\boldsymbol{j} + (v_1 w_2 - v_2 w_1)\boldsymbol{k}$$

ie. $\quad \boldsymbol{v} \times \boldsymbol{w} \;=\; \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \boldsymbol{k}$

can be evaluated by expanding above determinant along the first row.

8.1.6 Proof

This follows directly by expanding:

$$\boldsymbol{v} \times \boldsymbol{w} = (v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}) \times (w_1\boldsymbol{i} + w_2\boldsymbol{j} + w_3\boldsymbol{k})$$

$$= v_1\boldsymbol{i} \times (w_1\boldsymbol{i} + w_2\boldsymbol{j} + w_3\boldsymbol{k})$$
$$+ v_2\boldsymbol{j} \times (w_1\boldsymbol{i} + w_2\boldsymbol{j} + w_3\boldsymbol{k})$$
$$+ v_3\boldsymbol{k} \times (w_1\boldsymbol{i} + w_2\boldsymbol{j} + w_3\boldsymbol{k})$$

$$= v_1 w_2 (\boldsymbol{i} \times \boldsymbol{j}) - v_1 w_3 (\boldsymbol{k} \times \boldsymbol{i})$$
$$- v_2 w_1 (\boldsymbol{i} \times \boldsymbol{j}) + v_2 w_3 (\boldsymbol{j} \times \boldsymbol{k})$$
$$+ v_3 w_1 (\boldsymbol{k} \times \boldsymbol{i}) - v_3 w_2 (\boldsymbol{j} \times \boldsymbol{k})$$

$$= (v_2 w_3 - v_3 w_2)\boldsymbol{i} - (v_1 w_3 - v_3 w_1)\boldsymbol{j} + (v_1 w_2 - v_2 w_1)\boldsymbol{k}.$$

8.1.7 Example

Let $\boldsymbol{v} = (1, 2, 3)$, $\boldsymbol{w} = (4, 5, 6)$. Find a (non-zero) vector orthogonal to both $\boldsymbol{v}$ and $\boldsymbol{w}$.

The vector $\boldsymbol{v} \times \boldsymbol{w}$ will do:

$$
\begin{aligned}
\boldsymbol{v} \times \boldsymbol{w} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} \\
&= \boldsymbol{i} \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} - \boldsymbol{j} \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} + \boldsymbol{k} \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \\
&= \boldsymbol{i}(12 - 15) - \boldsymbol{j}(6 - 12) + \boldsymbol{k}(5 - 8) \\
&= -3\boldsymbol{i} + 6\boldsymbol{j} - 3\boldsymbol{k}.
\end{aligned}
$$

This is orthogonal to both $\boldsymbol{v}$ and $\boldsymbol{w}$.

8.2 Application: area of a triangle

Find area of the ΔABC with vectors $\boldsymbol{v}$, $\boldsymbol{w}$ along edges as shown in Figure 13.
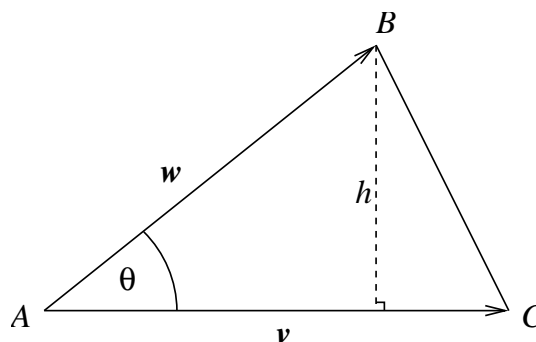

Figure 13: Find the area of the triangle ABC

$$
\begin{aligned}
\text{Area} &= \frac{1}{2} \text{ base} \cdot \text{perpendicular height} \\
&= \frac{1}{2} \|\boldsymbol{v}\| \cdot \|\boldsymbol{w}\| \sin \theta = \frac{1}{2} \|\boldsymbol{v} \times \boldsymbol{w}\|.
\end{aligned}
$$

8.2.1 Example

Find the area of a triangle with vertices $A = (0, 1, 3)$, $B = (1, 2, 1)$, $C = (4, 1, 0)$ and obtain a vector perpendicular to the plane of $\triangle ABC$.

$$\boldsymbol{w} = \overrightarrow{AB} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \quad \boldsymbol{v} = \overrightarrow{AC} = \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$$

Hence a vector perpendicular to the plane of $\triangle ABC$ is

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 4 & 0 & -3 \\ 1 & 1 & -2 \end{vmatrix} = 3\boldsymbol{i} + 5\boldsymbol{j} + 4\boldsymbol{k}.$$

Area of the triangle

$$
\begin{aligned}
\frac{1}{2} \|\boldsymbol{v} \times \boldsymbol{w}\| &= \frac{1}{2} \|3\boldsymbol{i} + 5\boldsymbol{j} + 4\boldsymbol{k}\| \\
&= \frac{1}{2}\sqrt{9 + 25 + 16} \\
&= \frac{1}{2}\sqrt{50} \\
&= \frac{5\sqrt{2}}{2}.
\end{aligned}
$$

8.3   Scalar Triple Product

The *number* $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$ is called the *scalar triple product* of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$.

Using the result of the previous theorem, express this product in terms of components.

Set $\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}$. Then

$$
\begin{aligned}
\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) &= (u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}) \cdot \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \boldsymbol{k} \right) \\[2mm]
&= u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \\[2mm]
&= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.
\end{aligned}
$$

This is a 3×3 determinant.

8.4   Geometrical Interpretation

The volume V of a parallelepiped determined by vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ along edges as shown is given by

$$
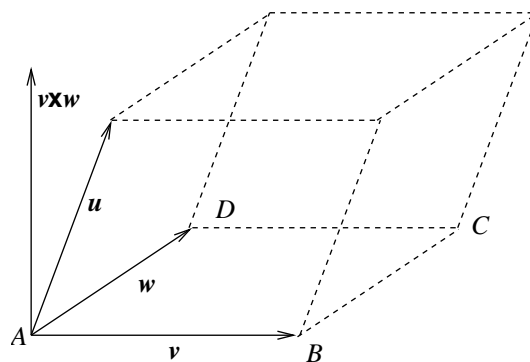V = |\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})|
$$

Figure 14: Parallelepiped determined using vectors.

8.4.1 Proof

$V = (\text{area of base } ABCD) \times h$ where

$$
\begin{aligned}
h \; &= \; \text{perpendicular height} \\
&= \; \text{size of component of } \boldsymbol{u} \text{ in direction of } \boldsymbol{v} \times \boldsymbol{w} \\
&= \; \left| \frac{\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})}{\| \boldsymbol{v} \times \boldsymbol{w} \|^2} (\boldsymbol{v} \times \boldsymbol{w}) \right| = \frac{|\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})|}{\| \boldsymbol{v} \times \boldsymbol{w} \|}
\end{aligned}
$$

From previous example,

$$
\begin{aligned}
\text{area of the base } ABCD \; &= \; 2 \times (\text{area } \Delta\, ABD) \\
&= \; \| \boldsymbol{v} \times \boldsymbol{w} \|
\end{aligned}
$$

$$
\Rightarrow \quad V = \| \boldsymbol{v} \times \boldsymbol{w} \| \cdot h = |\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})| .
$$

8.4.2 Example

Find the volume V of the parallelepiped determined by the vectors

$$
\boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad , \quad \boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad , \quad \boldsymbol{c} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} .
$$

$V = |\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})|$, where

$$
\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) \; = \; \begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1 - 2 = -1 \quad \text{(expanded along } r_3\text{)}
$$

$$
\Rightarrow \quad V \; = \; |-1| = 1.
$$

The volume V of a parallepiped determined by the vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ vanishes iff vectors are co-planar; therefore vectors are co-planar iff

$$
\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0 \quad - \quad \begin{array}{l} \text{this gives a useful method for determining} \\ \text{when 3 vectors are co-planar.} \end{array}
$$

9 Eigenvalues and eigenvectors

Let A be an $n \times n$ matrix and consider the equation

$$A\boldsymbol{x} = \lambda\boldsymbol{x} \tag{9.1}$$

with λ a number (possibly complex). It is clear that $\boldsymbol{x} = \boldsymbol{0}$ is always a solution to (9.1). A value λ, for which (9.1) has solution $\boldsymbol{x} \neq \boldsymbol{0}$ is called an *eigenvalue* of A. The corresponding solution vector $\boldsymbol{x} \neq \boldsymbol{0}$ is called an *eigenvector* of A corresponding to the eigenvalue λ.

9.1 Eigenvalues

9.1.1 Example

Let
$$A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix},$$

One eigenvalue is $\lambda = -2$ with corresponding eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

since $\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} = -2 \begin{pmatrix} -1 \\ 1 \end{pmatrix},$

another eigenvalue is $\lambda = 5$ with corresponding eigenvector $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

9.1.2 Note

Usually $A\boldsymbol{x}$ is related to $\boldsymbol{x}$ in a much more complicated way,

$$\text{e.g.} \quad \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \end{pmatrix} \neq \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

9.2 Eigenspace

9.2.1 Proposition (eigenspace is a vector space)

Let λ be an eigenvalue of A. Then the set of solutions to (9.1) forms a vector space called the *eigenspace* of A corresponding to eigenvalue λ, denoted N_λ. So $N_\lambda = \{x \in \mathbb{R}^n : Ax = \lambda x\}$.

9.2.2 Proof

(o) Note that $\mathbf{0}$ is always a solution to (9.1), so $\mathbf{0} \in N_\lambda$.

Take $x, y \in N_\lambda$, $\alpha \in \mathbb{R}$.

(i) Then

$$
\begin{aligned}
A(x + y) &= Ax + Ay \\
&= \lambda x + \lambda y \\
&= \lambda(x + y) \\
\Rightarrow\quad & x + y \in N_\lambda.
\end{aligned}
$$

(ii)

$$
\begin{aligned}
A(\alpha x) &= \alpha A x \\
&= \alpha \lambda x \\
&= \lambda(\alpha x) \\
\Rightarrow\quad & \alpha x \in N_\lambda.
\end{aligned}
$$

Hence the set of solutions to (9.1) forms a vector space.

9.3 How to find eigenvalues

We have
$$
\begin{aligned}
Ax &= \lambda x \\
\iff\quad Ax &= \lambda(Ix) \\
\iff\quad Ax - \lambda I x &= \mathbf{0} \\
\iff\quad (A - \lambda I)x &= \mathbf{0}.
\end{aligned}
$$
We seek a non-zero solution x of this equation. Recall that an $n \times n$ system $Bx = \mathbf{0}$ has a solution $x \neq \mathbf{0}$ iff B is not invertible iff $|B| = 0$.

Thus A has eigenvalue λ if and only if

$$|A - \lambda I| = 0.$$

9.3.1 Definition: Characteristic polynomial, characteristic equation

The determinant
$$p(\lambda) = |A - \lambda I|$$
is a polynomial of degree n in λ called the *characteristic polynomial* of A. We call equation (9.2) the *characteristic equation.*

The eigenvalues of A are thus given by the roots of the characteristic equation (9.2). Note that this means that A has at least 1 and at most n distinct eigenvalues (possibly complex). The corresponding eigenvectors may then be obtained for each λ in turn by solving

$$(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$$

using Gaussian elimination.

9.3.2 Example

Find the eigenvalues and eigenvectors of
$$A = \begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix}.$$

We have
$$A - \lambda I = \begin{pmatrix} -5 - \lambda & 2 \\ 2 & -2 - \lambda \end{pmatrix}.$$
So
$$p(\lambda) = |A - \lambda I| = \begin{vmatrix} -5 - \lambda & 2 \\ 2 & -2 - \lambda \end{vmatrix} = (\lambda + 5)(\lambda + 2) - 4$$

$$= \lambda^2 + 7\lambda + 6 = (\lambda + 6)(\lambda + 1).$$
Solving $p(\lambda) = 0$ yields the eigenvalues $\lambda_1 = -1$, $\lambda_2 = -6$.

To get the corresponding eigenvectors, solve $(A - \lambda_i)\boldsymbol{x} = \boldsymbol{0}$ for $i = 1, 2$.

$$\lambda_1 = -1: \quad \left(\begin{array}{cc|c} -4 & 2 & 0 \\ 2 & -1 & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 2 & -1 & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array} \right)$$

with general solution

$$\boldsymbol{x} = \begin{pmatrix} \alpha \\ 2\alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ is an eigenvector corresponding}$$

$$\text{to the eigenvalue } \lambda = -1;$$

$$\lambda_2 = -6: \quad \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

$$\Rightarrow \text{ general solution } \boldsymbol{x} = \begin{pmatrix} -2\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Hence $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = -6$.

Note: for each eigenvalue λ there will be a whole vector space of solutions. In fact $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is a basis for the one-dimensional eigenspace N_{-1}.

9.3.3   Example

Find eigenvalues and eigenvectors of $A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$.

$$
\begin{aligned}
p(\lambda) &= |A - \lambda I| \\[2mm]
&= \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} \\[2mm]
&= -(\lambda+2)\begin{vmatrix} 1-\lambda & -6 \\ -2 & -\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & -6 \\ -1 & -\lambda \end{vmatrix} - 3\begin{vmatrix} 2 & 1-\lambda \\ -1 & -2 \end{vmatrix} \\[2mm]
&= -(\lambda+2)(\lambda^2 - \lambda - 12) - 2(-2\lambda - 6) - 3(-\lambda - 3) \\[2mm]
&= -(\lambda+2)(\lambda+3)(\lambda-4) + 7(\lambda+3) \\[2mm]
&= (\lambda+3)[7 - (\lambda+2)(\lambda-4)] \\[2mm]
&= -(\lambda+3)(\lambda^2 - 2\lambda - 15) = -(\lambda+3)^2(\lambda-5).
\end{aligned}
$$

Solving $p(\lambda) = 0$ yields the eigenvalues $\lambda = -3, 5$.
To find eigenvectors corresponding to $\lambda = -3, 5$ we need to solve $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$. The augmented matrix $(A - \lambda I \mid \boldsymbol{0})$ for $\lambda = -3$ is

$$
\begin{pmatrix} 1 & 2 & -3 & | & 0 \\ 2 & 4 & -6 & | & 0 \\ -1 & -2 & 3 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -3 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}
$$

$$
\begin{aligned}
&\Rightarrow x_1 + 2x_2 - 3x_3 = 0 \\
&\quad x_1 = -2x_2 + 3x_3 \\
&\quad x_2 = \alpha \\
&\quad x_3 = \beta \\
&\Rightarrow \boldsymbol{x} = \begin{pmatrix} -2\alpha + 3\beta \\ \alpha \\ \beta \end{pmatrix} = \alpha\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}
\end{aligned}
$$

is the general solution.

So $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ are linearly independent eigenvectors for $\lambda_1 = -3$, and they form a basis of the eigenspace.

For $\lambda_2 = 5$ the augmented matrix is

$$\left(\begin{array}{ccc|c} -7 & 2 & -3 & 0 \\ 2 & -4 & -6 & 0 \\ -1 & -2 & -5 & 0 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 2 & -4 & -6 & 0 \\ -7 & 2 & -3 & 0 \end{array} \right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 0 & -8 & -16 & 0 \\ 0 & 16 & 32 & 0 \end{array} \right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

So:

$$
\begin{aligned}
x_1 + 2x_2 + 5x_3 &= 0 & \Rightarrow x_1 &= -2x_2 - 5x_3 \\
x_2 + 2x_3 &= 0 & \Rightarrow x_2 &= -2x_3 \\
x_3 &= \alpha \\
\Rightarrow x_1 &= 4\alpha - 5\alpha = -\alpha \\
x_2 &= -2\alpha, & x_3 &= \alpha.
\end{aligned}
$$

The vector $\boldsymbol{x} = \begin{pmatrix} -\alpha \\ -2\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$ is the general solution,

so $\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue $\lambda = 5$.

9.3.4 Example

Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$.

$p(\lambda) = |A - \lambda I| = \begin{vmatrix} -\lambda & 2 \\ -2 & -\lambda \end{vmatrix} = \lambda^2 + 4.$

Eigenvalues: $p(\lambda) = 0 \Rightarrow \lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i \Rightarrow \lambda_1 = 2i; \; \lambda_2 = -2i.$

In this case A has two complex eigenvalues $\lambda = \pm 2i$. To obtain eigenvectors solve $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$, $\lambda = \pm 2i$, exactly as before, using complex numbers. The augmented matrix $(A - \lambda I \mid \boldsymbol{0})$ corresponding to $\lambda_1 = 2i$ is

$$\left(\begin{array}{cc|c} -2i & 2 & 0 \\ -2 & -2i & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} -2i & 2 & 0 \\ 0 & 0 & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array} \right)$$

$$\Rightarrow \quad x_1 + ix_2 = 0 \quad \Rightarrow \quad x_1 = -ix_2 = -i\alpha$$
$$x_2 = \alpha$$

$$\Rightarrow \boldsymbol{x} = \begin{pmatrix} -i\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} -i \\ 1 \end{pmatrix} \text{ is the general solution,}$$

and $\begin{pmatrix} -i \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_1 = 2i$.

For $\lambda = -2i$:

$$\left(\begin{array}{cc|c} 2i & 2 & 0 \\ -2 & 2i & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array} \right)$$

$$\Rightarrow \quad x_1 - ix_2 = 0 \Rightarrow x_1 = ix_2 = i\alpha$$
$$x_2 = \alpha$$

$$\Rightarrow \quad x = \begin{pmatrix} i\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} i \\ 1 \end{pmatrix} \text{ is the general solution, and } \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$\text{is an eigenvector for } \lambda = -2i.$$

9.3.5 Note

Solving the system $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ to find eigenvectors always leads to at least one row of zeros when $A - \lambda I$ is row reduced. This is because λ is chosen so that $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ has a solution $\boldsymbol{x} \neq \boldsymbol{0}$.

$\therefore A - \lambda I$ is not invertible. $\therefore A - \lambda I$ does not reduce to I.

9.3.6 A note on applications

Eigenvalues and eigenvectors have many applications e.g. in Quantum Mechanics the energy of a system is represented by a matrix whose eigenvalues give the allowed energies. This is why discrete energy levels are observed. Another situation where they are used is in modelling the population dynamics of a predator-prey system in biology. The eigenvalues and eigenvectors determine the stability and dynamics of a population in such models.

There are many other applications in Mathematics, Physics, Chemistry and Biology.

10 Numbers

Mathematics uses structures at a fundamental level. These basic structures include numbers, sets (e.g. intervals), shapes, vectors, complex numbers, quaternions and many more strange and exotic things. Numbers are very important in mathematics as building blocks as they lead to equations and inequalities which we can use as a tool to solve many abstract and applied problems.

10.1 Number systems

$\mathbb{R}$: set of real numbers - represented by a finite or infinite decimal expansion

The following are commonly used subsets of $\mathbb{R}$:

$\mathbb{N}$: set of natural numbers $\{1, 2, 3, \ldots\}$

$\mathbb{Z}$: set of integers $\{0, 1, -1, 2, -2, 3, -3, \ldots\}$

$\mathbb{Q}$: set of rational numbers - of the form $\dfrac{p}{q}$ where $p, q \in \mathbb{Z}$, $q \neq 0$

Irrational numbers are real numbers which cannot be represented as a ratio of integers. For example, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\pi = 3.14159\ldots$, $e = 2.71828\ldots$ are all irrational. Note: proving rationality or irrationality of a given number can be quite subtle!

10.2 Real number line and ordering on $\mathbb{R}$

The real number system can be visualised by imagining each real number as a point on a line, with the positive direction being to the right, and an arbitrary origin being chosen to represent 0 (see Figure 15).
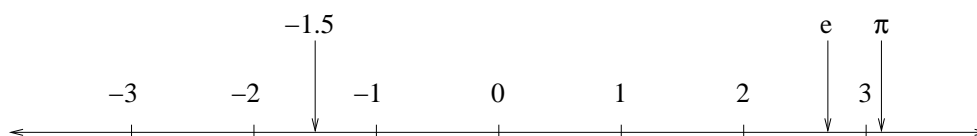

Figure 15: The real number line.

The real numbers are *ordered*, i.e. given any 2 real numbers a and b there holds *precisely* one of the following: $a > b$, $a < b$ or $a = b$. This means we can use the symbols '<', '>', '≤' and '≥' to write statements such as $1 \leq 2$ and $3 > \sqrt{3}$. Geometrically, $a < b$ means that a lies to the left of b on the real number line. Note also that $a \leq b$ means that either $a < b$ or $a = b$.

10.3 Definition: Intervals

An interval is a set of real numbers that can be thought of as a segment of the real number line. For $a < b$, the *open interval* from a to b is given by

$$(a, b) = \{x \in \mathbb{R}| \ a < x < b\}.$$

This notation is interpreted as

(a, b) is the set of all real numbers x such that $a < x < b$.

Note here that the end points are *not* included. If we wanted to include the endpoints, we would have a *closed interval* from a to b, denoted

$$[a, b] = \{x \in \mathbb{R}| \ a \leq x \leq b\}.$$

In this notation it is important to distinguish between the round brackets (a, b) for an open interval (excluding the end points) and the square brackets $[a, b]$ for a closed interval (including the end points).

We also have *half-open intervals* (or half-closed depending on how you feel), denoted

$$[a, b) = \{x \in \mathbb{R}| \ a \leq x < b\}, \text{ and}$$
$$(a, b] = \{x \in \mathbb{R}| \ a < x \leq b\}.$$

There are also semi-infinite intervals such as

$$(-\infty, a] = \{x \in \mathbb{R} \mid x \leq a\}, \text{ and}$$
$$(a, \infty) = \{x \in \mathbb{R} \mid a < x\}.$$

There are other possible types of semi-infinite intervals: Stewart, 6ed. p. A4; 5ed. p. A4.

10.4 Absolute value

We define
$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

10.4.1 Examples

$$\begin{aligned} |3| &= 3 \\ |-4| &= 4 \\ |1 - \pi| &= \pi - 1 \end{aligned}$$

10.4.2 Properties of absolute value

(i) $|a|^2 = a^2 \implies |a| = \sqrt{a^2}$ (positive square root)

(ii) $|ab| = |a||b|$

(iii) $|a + b| \leq |a| + |b|$ (triangle inequality)

(iv) $|a^n| = |a|^n$, $n \in \mathbb{Z}$, with $a \neq 0$ for nonpositive n.

It is worth emphasising that in general $|a + b| \neq |a| + |b|$, but the triangle inequality does hold:
$$|a + b| \leq |a| + |b|.$$

For example, set $a = 1$ and $b = -2$. We have l.h.s $= |1 + (-2)| = |-1| = 1$, r.h.s $= |1| + |-2| = 1 + 2 = 3$.

10.4.3 Convention for $\sqrt{}$

For $a > 0$, $\sqrt{a}$ always denotes the positive square root. Thus $\sqrt{4} = 2$ and so on. For $a > 0$, solutions to $x^2 = a$ are $x = \pm\sqrt{a}$.

Notes

Notes

11 Functions

11.1 Notation

Let X and Y be subsets of $\mathbb{R}$. The notation $f : X \to Y$ means "f maps from X into Y".

11.2 Definition: Function, domain, range

A function $f : X \to Y$ is a rule which assigns to every element $x \in X$ *exactly one* element $f(x) \in Y$ called the value of f at x. X is called the *domain* of f and

$$f(X) = \{f(x)| \ x \in X\}$$

is called the *range* of f.

The range of f, $f(X)$, is a subset of Y. In other words, you can take any x value from X, plug it into f, and return the value $f(x)$, which is in Y. The range is the set of all possible values of $f(x)$ as x varies throughout the domain. Note that $f(X)$ is not necessarily equal to all of Y.

11.2.1 Example

The function $f : \mathbb{R} \to \mathbb{R}$, such that $f(x) = x^2$ is a function with domain $\mathbb{R}$. The range is given by

$$
\begin{aligned}
f(\mathbb{R}) \ &= \ \{f(x) \mid \ x \in \mathbb{R}\} \\
\\
&= \ \{x^2 \mid \ x \in \mathbb{R}\} \\
\\
&= \ \{y \in \mathbb{R} \mid \ y \geq 0\} \ = \ [0, \infty) \subseteq \mathbb{R}
\end{aligned}
$$

11.2.2 Example

The following is a perfectly good example of a function, $g : (-6, 7) \to \mathbb{R}$

$$
g(x) = \begin{cases} -5 &, & -6 < x < 0 \\ -\pi &, & x = 0 \\ x &, & 0 < x < 7 \end{cases}
$$

Piecewise defined functions are discussed in Stewart, 6ed. p. 17; 5ed. p. 18.

Clearly the domain of g is the open interval $(-6, 7)$, but what about the range? Looking at the algebraic expression for g given above, the range can be expressed as

$$g((-6, 7)) = \{-5\} \cup \{-\pi\} \cup (0, 7).$$

11.3 Graphs

We can represent a function by drawing its *graph* which is the set of all points (x, y) in a plane where $y = f(x)$.

11.3.1 Example

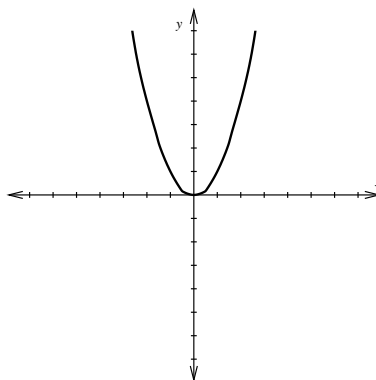Let's plot the previous two functions:

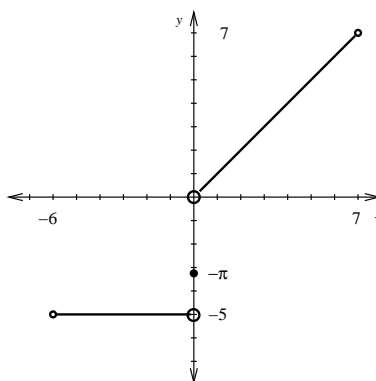Figure 16: Graph of $f(x) = x^2$ from Example 11.2.1

Figure 17: Graph of $g(x)$ from Example 11.2.2.

11.3.2   Example

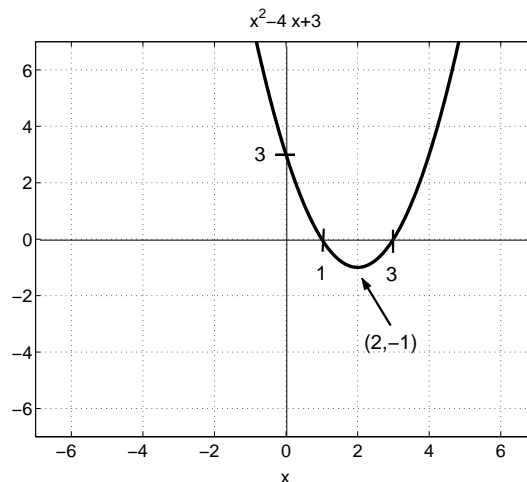Graph the function $f(x) = x^2 - 4x + 3$ on the axes in Figure 18.

Figure 18: $f(x) = x^2 - 4x + 3 = (x-1)(x-3) = (x-2)^2 - 1$

11.4   Convention (domain)

An expression like "the function $y = \sqrt{1 - x^2}$" means the function f with $y = f(x) = \sqrt{1 - x^2}$. When the domain is not specified it is taken to be the largest set on which the rule is defined. In this example, the domain would be $[-1, 1]$.

11.5   Caution (non-functions)

Sometimes we see equations for curves which are not functions. Consider the equation of a circle of radius 1, centred at the origin: $x^2 + y^2 = 1$. The crucial function property states that for each value x in the domain there must correspond *exactly one* value y in the range. If there are any more y values, then it is not a function.

11.6 Vertical line test

In the graph of a function, the vertical line $x =$ constant must cut the graph in *at most one point*. The graph for the circle $x^2 + y^2 = 1$ is given in Figure 19. We can clearly see that the vertical line intersects the circle at two points. In this case the two y values are given by $y = \pm\sqrt{1 - x^2}$. Therefore $x^2 + y^2 = 1$ does not give rise to a function on any domain intersecting $(-1, 1)$.
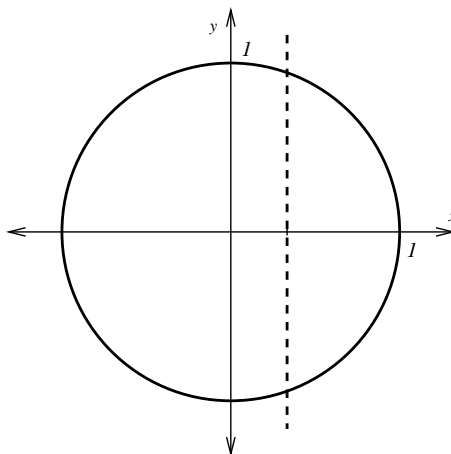

Figure 19: The circle $x^2 + y^2 = 1$ with a vertical line intersecting at two points. This shows that the equation for a circle does not give rise to a function.
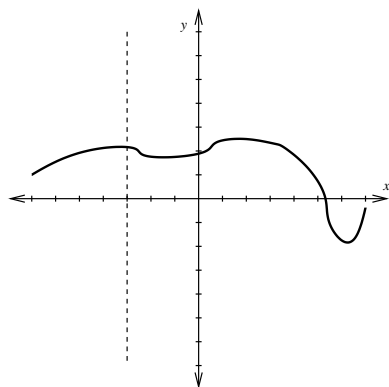
11.6.1 Example

Both $y = \sqrt{1 - x^2}$ (the top half of the circle) and $y = -\sqrt{1 - x^2}$ (the bottom half of the circle) are functions of x. What are the domains and ranges of these two functions?
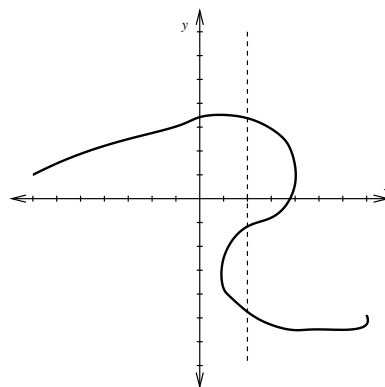
Put $f(x) = \sqrt{1 - x^2}$; $g(x) = -\sqrt{1 - x^2}$.
Then dom$(f) =$ dom$(g) = [-1, 1]$,
and range$(f) = [0,1]$; range$(g) = [-1, 0]$.

11.6.2 Example

Look at the graphs in Figures 20(a) and 20(b). Do either of these graphs represent functions?

(a) (b)

Figure 20: Do either of these two graphs represent functions?

a) Yes.
b) Not on the entire interval.

11.7 Exponential functions

An exponential function is one of the form $f(x) = a^x$, where the *base a* is a positive constant, and x is said to be the *exponent* or *power*. One very common exponential function which we shall see often in this course is given by $f(x) = e^x$. It has the graph shown in Figure 21. Notice that it cuts the y-axis (the line $x = 0$) at $y = 1$. This graph was made using Matlab with the command `ezplot('exp(x)',[-1.75,1.75,-4,6])`.
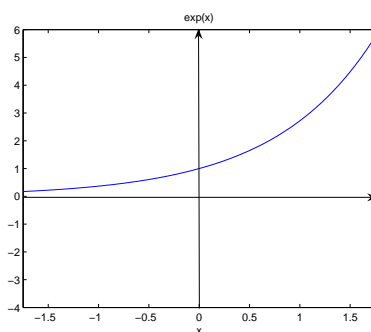

Figure 21: The function $f(x) = e^x$.

11.7.1 Properties of exponents

For $a > 0$ and $x, y, r \in \mathbb{R}$:

$$\begin{aligned}
\text{(i)} \quad & a^x \cdot a^y = a^{x+y} \\
\text{(ii)} \quad & a^x / a^y = a^{x-y} \\
\text{(iii)} \quad & a^{xr} = (a^x)^r
\end{aligned}$$

Exponential functions are very useful for modelling many natural phenomena such as population growth (base $a > 1$) and radioactive decay (base $0 < a < 1$).

11.7.2 Example: Stewart, 5ed. p. 426

The half-life of the isotope strontium-90, ^{90}Sr, is 25 years. This means that half of *any* quantity of ^{90}Sr will disintegrate in 25 years. Say the initial mass of a sample is 24mg, write an expression for the mass remaining after t years.

Let $m(t)$ be the mass remaining at time t, and $m(0) = m_0$.
Find $m(t)$ as a function of t.
$h = $ half-life: $m(h) = \dfrac{m_0}{2}$

So $m(2h) = \frac{m_0}{4}$, $m(3h) = \frac{m_0}{8}$, ..., $m(nh) = \frac{m_0}{2^n}$.

Hence $m(xh) = \frac{m_0}{2^x} = m_0 \cdot 2^{-x}$, for $x \in \mathbb{R}$, $x \geq 0$.

So $m(t) = m(\dfrac{t}{h}h) = \dfrac{m_0}{2^{t/h}} = m_0 \cdot 2^{-t/h}$.
For the current problem, $m_0 = 24$mg, $h = 25$ years, so $m(t) = 24 \cdot 2^{-t/25}$mg.

In general, if the rate of change of a function is proportional to the value of the function, that is

$$\frac{df(t)}{dt} = kf(t),$$

then

$$f(t) = f(0)e^{kt},$$

where $f(0)$ is the initial value of the function and k is the rate of exponential growth or decay.

11.7.3   Example

Suppose that there are 1000 rabbits frollicking in your expansive yard on January 1, 2006. Thirty days later there are 1500 rabbits. Find an expression for the rabbit population and then determine how many rabbits there will be 60 days after January 1, 2006.

Let $n(t)$ be the number of rabbits at time t. Then $n(t) = n(0)e^{kt}$, where $n(0) = 1000$. We now use the fact that $n(30) = 1500$ to solve for the rate of exponential growth, k.

$$\begin{aligned} n(30) = 1500 &= 1000e^{30k}, \\ \frac{1500}{1000} &= e^{30k}. \end{aligned}$$

Hence (see §11.13):

$$\begin{aligned} \ln\left(\frac{3}{2}\right) &= 30k, \\ k &= \frac{1}{30}\ln\left(\frac{3}{2}\right), \\ &\approx 0.0135. \end{aligned}$$

Therefore, $n(t) = 1000e^{0.0135t}$, and $n(60) = 2250$.

11.8   Trigonometric functions (sin, cos, tan)

In calculus, we measure angles in *radians*. One radian is defined as the angle subtended at the centre of a circle of radius 1 by a segment of arc length 1.
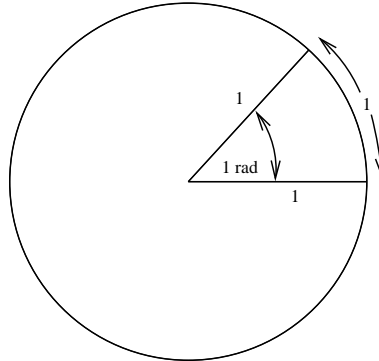

Figure 22: Angle in radians.

By definition, 2π radians $= 360°$. To convert between the two measures use the following formulae:

For x angle in radians, θ angle in degrees,

$$x \times \frac{360°}{2\pi} = \theta, \quad \theta \times \frac{2\pi}{360°} = x.$$

The point on the unit circle $x^2 + y^2 = 1$ making an angle θ (in radians) anti-clockwise from the x-axis has coordinates $(\cos\theta, \sin\theta)$, so $\cos^2\theta + \sin^2\theta = 1$. This defines the cos and sin functions. This relationship is shown in Figure 23.

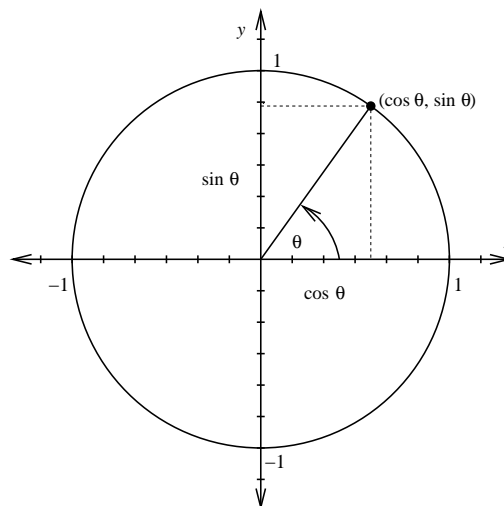Figure 23: The relationship between the cosine and sine functions.

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The graphs of the functions $\cos\theta$ and $\sin\theta$ are shown below in Figure 24. Note how the two functions have the same behaviour, but are shifted (i.e. *phase shifted)*) by $\frac{\pi}{2}$. This leads to the relationship

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right) = \sin\left(\theta + \frac{\pi}{2}\right).$$
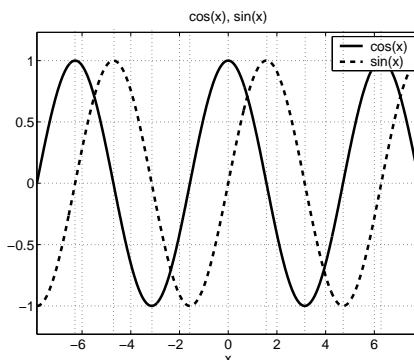
Figure 24: The relationship between the cosine and sine functions.

In both cases in Figure 24 the range is $[-1, 1]$. There are some unmarked vertical dotted lines. What values do they represent? (This graph was made using Matlab. To get both curves on the same axes, we used the `hold on` command.)

The tan function is related to the sine and cosine functions by

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

It is defined for $\cos\theta \neq 0$. It is therefore not defined at values of θ which are odd integer multiples of $\frac{\pi}{2}$ (eg $\pm\frac{\pi}{2}$, $\pm\frac{3\pi}{2}$, $\pm\frac{5\pi}{2}$ etc).

The graph of $\tan\theta$ is given in Figure 25 below. Mark the x values in the graph where the function is not defined.
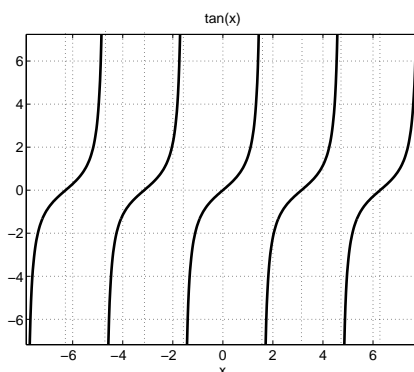
Figure 25: Graph of $\tan\theta$.

Figure 26 shows another common function,

$$\cot x = \frac{\cos x}{\sin x}.$$
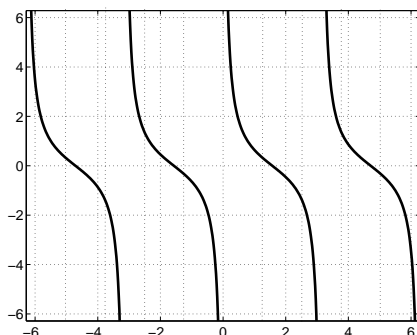
Figure 26: Graph of $\cot \theta$.

The periodic nature of all the trigonometric functions we have seen makes them ideal for modelling repetitive phenomena such as tides, vibrating strings, and various types of natural wave-like behaviour.

Both the sine and cosine functions have *period* 2π. Hence

$$\sin(x + 2\pi) = \sin x, \text{ and}$$
$$\cos(x + 2\pi) = \cos x.$$

Furthermore, both $\sin(x)$ and $\cos(x)$ have an *amplitude* of one.

We can "speed up" or "slow down" these functions by tinkering with their periodicity. For example, compare the functions $\sin(x)$ and $\sin\left(\frac{x}{2}\right)$ in Figure 27. Compare also the functions $\sin(x)$ and $\sin(2x)$ in Figure 28.
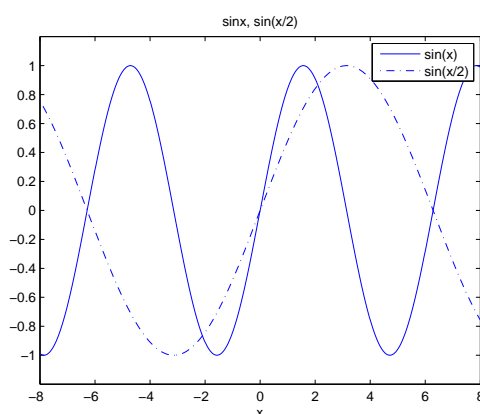
Figure 27: Graph of $\sin(x)$ and $\sin\left(\frac{x}{2}\right)$.
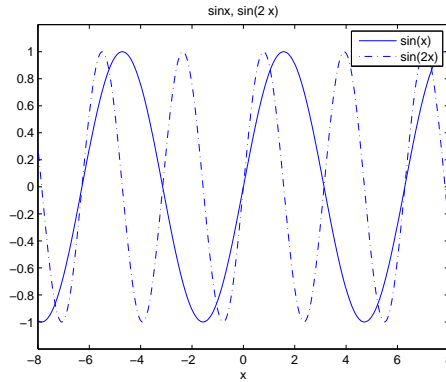
Figure 28: Graph of $\sin(x)$ and $\sin(2x)$.

In addition, we can stretch or shrink trigonometric functions by multiplying these functions by constants other than one. For example, Figure 29 shows the functions $\frac{1}{2}\cos(x)$ and $3\cos(x)$. What amplitudes do these have?
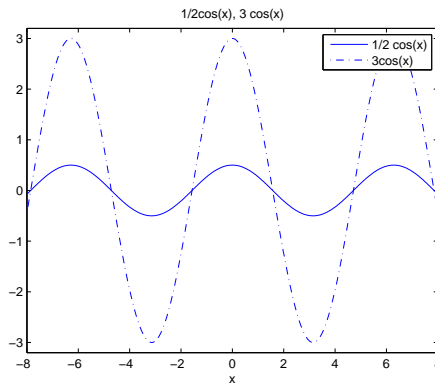
Figure 29: Graph of $\frac{1}{2}\cos x$ and $3\cos(x)$.

Naturally, we can change both the period and amplitude of trigonometric functions simultaneously. Figure 30 show the graphs of $4\cos\left(\frac{x}{3}\right)$ and $2\sin(5x)$.
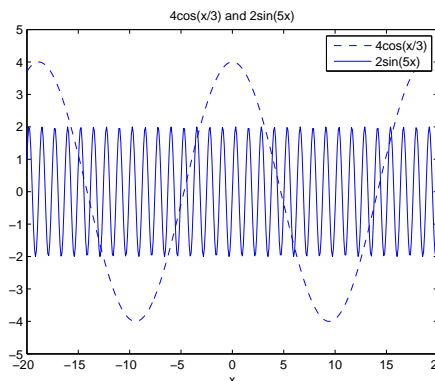
Figure 30: Graphs of $4\cos\left(\frac{x}{3}\right)$ and $2\sin(5x)$.

11.8.1 Example

Suppose that the temperature in Brisbane is periodic, peaking at 32 degrees Celsius on January 1, and sinking to 8 degrees Celsius in June; see Figure 31. Use the cosine function to model temperature as a function of time (in days).
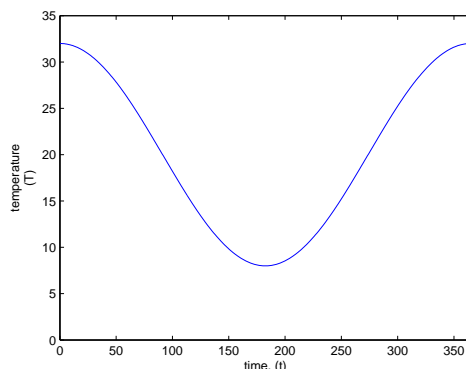
Figure 31: Graph of temperature fluctuations in Brisbane.

First note that the amplitude of this cosine function is $\frac{1}{2}(32 - 8) = 12$. Thus, we guess that our function will be of the form $T = 12\cos(t)$, where T is the temperature and t corresponds to time.

However, because we are measuring time in days the period of our model should be 365 days, while the period of $T = 12\cos(t)$ is 2π. We take care of this by horizontally scaling our function to give $T = 12\cos\left(\frac{2\pi}{365}t\right)$.

Finally, we shift our model up by 20, because we want it to give temperatures between 8 and 32 degrees, not -12 and 12 degrees.

Thus, the function which describes Brisbane's temperature as a function of time (measured in days) is

$$T = 20 + 12\cos\left(\frac{2\pi}{365}t\right).$$

11.9   Composition of functions

Let f and g be two functions. Then the composition of f and g, denoted $f \circ g$, is the function defined by

$$(f \circ g)(x) = f(g(x)).$$

Note: $\operatorname{dom}(f \circ g) = \{x \in \operatorname{dom}(g) \mid g(x) \in \operatorname{dom}(f)\}$

11.9.1   Example

$f(x) = x^2 + 1$, $g(x) = \dfrac{1}{x}$. Their compositions $f \circ g$ and $g \circ f$ are given by

$$
\begin{aligned}
(f \circ g)(x) = f(g(x)) &= f\left(\frac{1}{x}\right) \\
&= \frac{1}{x^2} + 1, \quad x \neq 0, \\
(g \circ f)(x) = g(f(x)) &= g(x^2 + 1) \\
&= \frac{1}{x^2 + 1}, \quad x \in \mathbb{R}.
\end{aligned}
$$

Notice how these two composite functions are not equal. In general, $f \circ g \neq g \circ f$.

11.10   One-to-one (1-1) functions - Stewart, 6ed. pp. 385-388; 5ed. pp. 413-417

A function $f : X \to Y$ is said to be *one-to-one* (usually written as 1-1) if, for all $x_1, x_2 \in X$,

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

Equivalently:

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2).$$

On the graph $y = f(x)$, 1-1 means that any horizontal line $y =$ constant cuts through the curve in at most one place; see Figure 32.

(a)   (b)
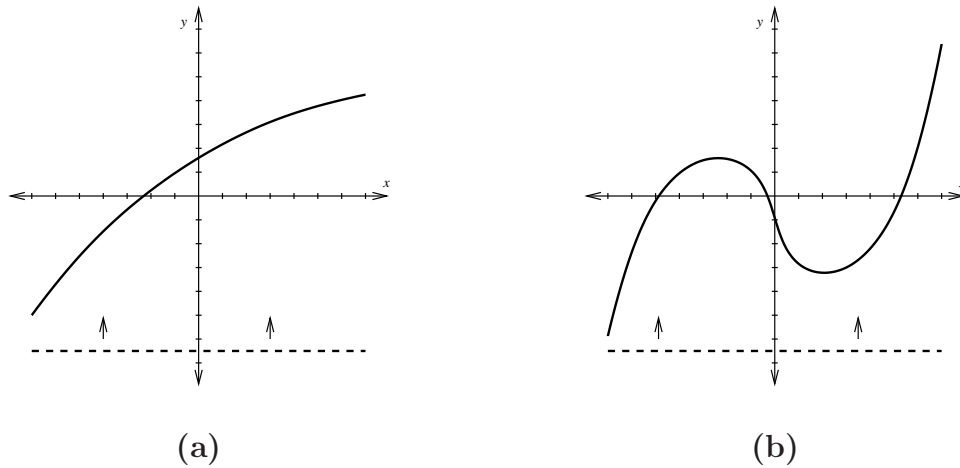
Figure 32: Are either of these functions 1-1?

a) Yes.
b) No.

11.10.1 Example

Show from the definition that the function $f(x) = e^{2x-5}$ is 1-1 on its domain.

The domain of the function is $\mathbb{R}$.

We need to show that

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \text{ for any } x_1, x_2 \in \mathbb{R}.$$

Let $x_1, \; x_2 \in \mathbb{R}$. Then

$$
\begin{aligned}
f(x_1) = f(x_2) \quad \Rightarrow \quad & e^{2x_1 - 5} = e^{2x_2 - 5} \\
\Rightarrow \quad & 2x_1 - 5 = 2x_2 - 5 \\
\Rightarrow \quad & 2x_1 = 2x_2 \\
\Rightarrow \quad & x_1 = x_2
\end{aligned}
$$

Therefore e^{2x-5} is 1-1.

11.11 Inverse Functions

Let $f : X \rightarrow Y$ be a 1-1 function. Then we can define the *inverse function*, $f^{-1} : f(X) \rightarrow X$ by

$$
\begin{aligned}
f^{-1}(f(x)) &= x, \quad \forall\, x \in X, \\
f(f^{-1}(y)) &= y, \quad \forall\, y \in f(X).
\end{aligned}
$$

By definition,

$$
\begin{aligned}
\text{domain } f^{-1} &= \text{range } f, \text{ and} \\[2mm]
\text{range } f^{-1} &= \text{domain } f.
\end{aligned}
$$

Note that $y = f(x) \Leftrightarrow x = f^{-1}(y)$, and that $(f^{-1}(y))^{-1} = f(x)$. The function f must be 1-1 in order that f^{-1} satisfy the crucial function property. In order to obtain the graph of $f^{-1}(x)$, we reflect the graph of $f(x)$ about the line $y = x$; for example, see Figure 33 below.
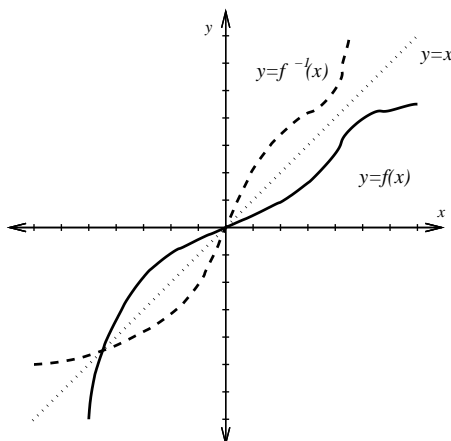
Figure 33: The dashed line represents the graph of the inverse function, having beenreflected about the line $y = x$.

11.11.1 Example

Draw the graph of the inverse function of $f(x) = x^3$ (the cube root function $f^{-1}(x) = x^{1/3}$) on the same axes given below in Figure 34.
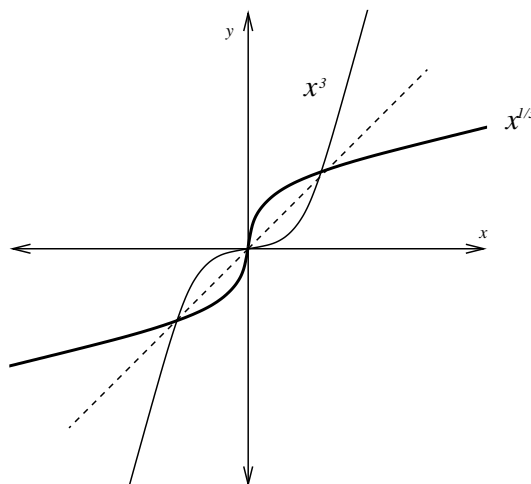
Figure 34: $f(x) = x^3$. Draw in the graph of $f^{-1}(x) = x^{1/3}$.

11.11.2 Example

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is not 1-1 and therefore has no inverse. However, the positive half $(x \geq 0)$ gives a 1-1 function $f : [0, \infty) \to \mathbb{R}$, $f(x) = x^2$ with range $[0, \infty)$. The inverse of this function is then $f^{-1} : [0, \infty) \to [0, \infty)$, $f^{-1}(x) = \sqrt{x}$. Similarly the negative half of the function $f(x) = x^2$ is 1-1, with inverse $f^{-1} : [0, \infty) \to (-\infty, 0]$, $f^{-1}(x) = -\sqrt{x}$.

This technique is often used when the function is not 1-1 over its entire domain: just take a part where it is 1-1 and determine the inverse for that part.

For further discussion on how to find the inverse function of a 1-1 function, see Stewart, 6ed. p. 388 box 5; 5ed. p. 416 box 5.

11.12   Logarithms - Stewart, 6ed. p. 405; 5ed. p. 434

Logarithms are the inverse functions of the exponential functions.
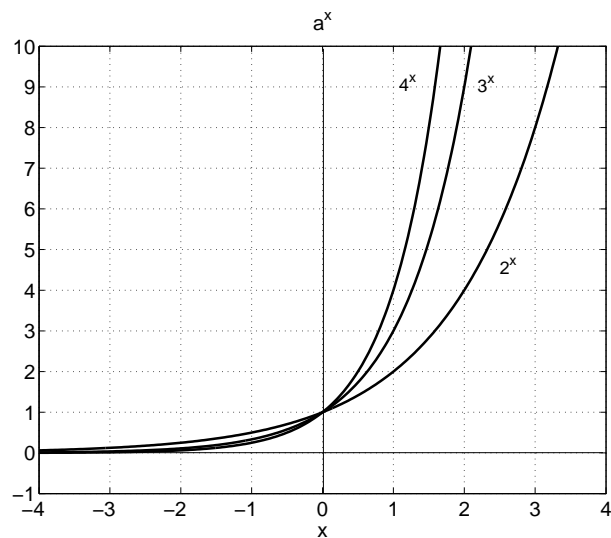
Figure 35: Three plots of $y = a^x$ with $a = 2, 3, 4$.

From the graph of $y = a^x$ ($a \neq 1$ a positive constant), we see that it is 1-1 and thus has an inverse, denoted $\log_a x$. From this definition we have the following facts:

$$\log_a(a^x) \;\; = \;\; x \quad \forall x \in \mathbb{R},$$

$$a^{\log_a x} \;\; = \;\; x \quad \forall x > 0.$$

What is the domain and range of $f(x) = \log_a(x)$?

The domain of $\log_a(x)$ is $(0, \infty)$ and its range is $\mathbb{R}$.

11.13   Natural logarithm

Now we set $a = e$ (Euler's number $= 2.71828\ldots$). The inverse function of $f(x) = e^x$ is

$$\log_e(x) \equiv \ln x.$$
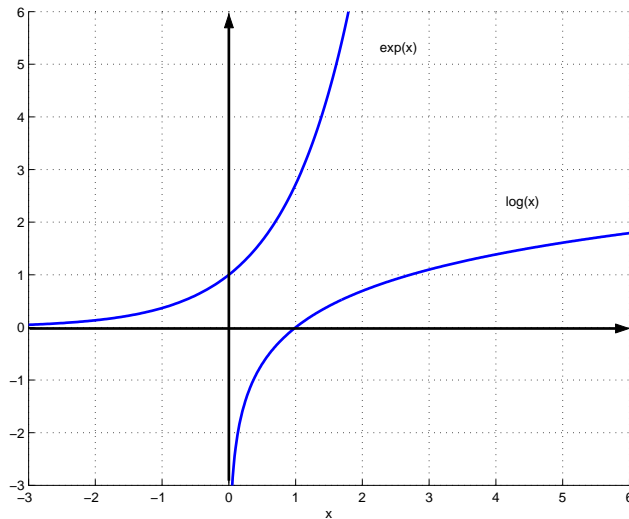
Figure 36: Graphs of $f(x) = e^x$ and $g(x) = \ln x$.

11.13.1 Properties

Using exponent laws, together with the fact that $\ln x$ is the inverse function of e^x, we can prove the following.

For $x, y > 0$, we have:

  (1) $\ln(xy) = \ln(x) + \ln(y)$;

  (2) $\ln\left(\dfrac{x}{y}\right) = \ln(x) - \ln(y)$; and

  (3) $\ln(x^r) = r \ln x$, $r \in \mathbb{R}$.

  (4) $\ln(1) = 0$

  (5) $\ln(e) = 1$

Note
$$\ln\left(\frac{1}{x}\right) = \ln\left(x^{-1}\right) = -\ln x.$$

Remark: The same laws hold for $\log_a$, except that (5) becomes $\log_a(a) = 1$.

11.13.2 Example: Bacteria population - Stewart, 6ed. p. 440 qn 49; 5ed. p. 440 qn 49

If a bacteria population starts with 100 bacteria and doubles every 3 hours, then the number of bacteria n after t hours is given by the formula

$$n = f(t) = 100 \cdot 2^{t/3}.$$

(a) Find the inverse of this function and explain its meaning.

(b) When will the population reach 50000?

(a) If $n = 100 \cdot 2^{t/3}$, then

$$\frac{n}{100} = 2^{t/3}.$$

Taking the natural logarithm of both sides gives

$$
\begin{aligned}
\ln\left(\frac{n}{100}\right) &= \ln\left(2^{t/3}\right) \\
&= \frac{t}{3}\ln(2) \quad \text{(using 11.13.1)}
\end{aligned}
$$

so that

$$t = 3 \cdot \frac{\ln(n/100)}{\ln 2}.$$

It describes how long it will take to obtain n bacteria.

(b) Substituting $n = 50,000$ gives $t \approx 26.9$ hours.

11.14   Inverse trigonometric functions

The function $y = \sin x$ is 1-1 if we just define it over the interval $[-\pi/2, \pi/2]$; see Figure 24. The inverse function for this part of $\sin x$ is denoted $\arcsin x$. Thus $\arcsin x$ is defined on the interval $[-1, 1]$ and takes values in the range $[-\pi/2, \pi/2]$. The graph can easily be obtained by reflecting the graph of $\sin x$ about the line $y = x$ over the appropriate interval; see Figure 37.
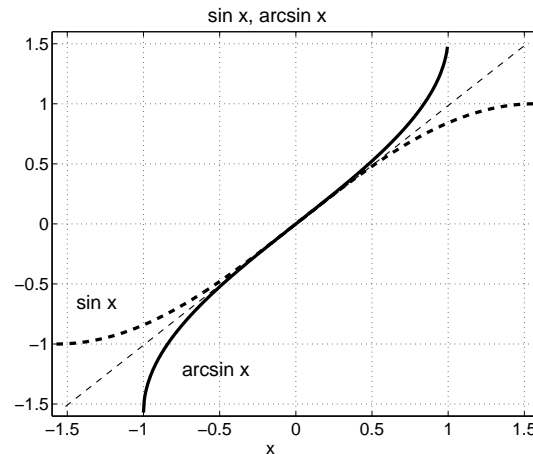
Figure 37: $f(x) = \sin x$ defined on $[-\pi/2, \pi/2]$ reflected about the line $y = x$ to give $f^{-1}(x) = \arcsin x$.

Similarly $y = \cos x$ is 1-1 on the interval $[0, \pi]$ and its inverse function is denoted $\arccos x$. The function $\arccos x$ is defined on $[-1, 1]$ and takes values in the range $[0, \pi]$. Figure 38 below shows the graphs of $f(x) = \cos x$ before and after reflection about the line $y = x$. This gives the graph of $f^{-1}(x) = \arccos x$.
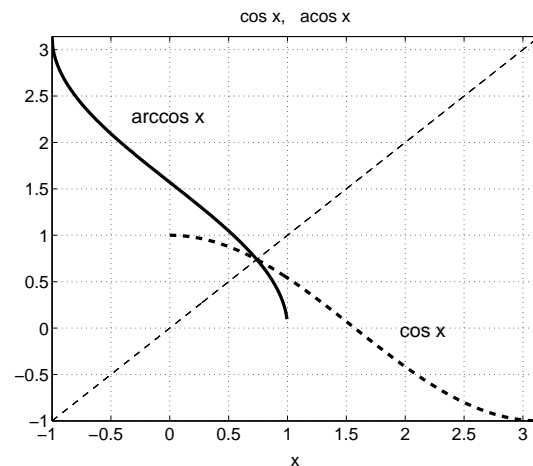
Figure 38: $f(x) = \cos x$ defined on $[0, \pi]$ reflected about the line $y = x$ to give $f^{-1}(x) = \arccos x$ (written in MATLAB as `acos(x)`).

Also, $\tan x$ is 1-1 on the open interval $(-\pi/2, \pi/2)$ with inverse function denoted by $\arctan x$. Hence arctan has the domain $(-\infty, \infty)$ with values in the range $(-\pi/2, \pi/2)$.
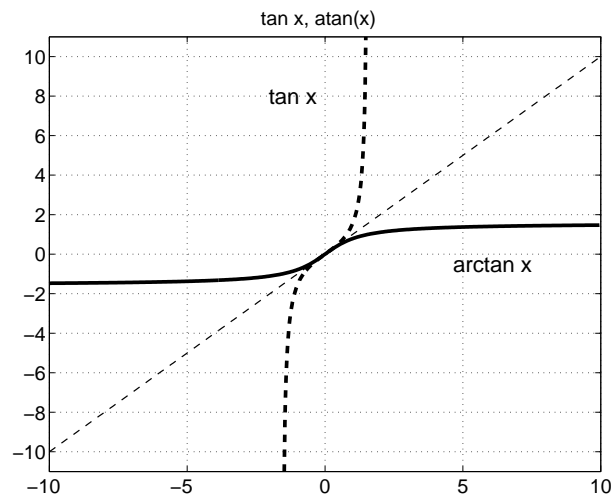
Figure 39: $f(x) = \tan x$.

Notes

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12 Limits

Limits arise when we want to find the tangent to a curve or the velocity of an object, for example. Once we understand limits, we can proceed to studying continuity and calculus in general. You should be aware that limits are a fundamental notion to calculus, so it is important to understand them clearly.

12.1 Definition: Limit - Stewart, 6ed. p. 66; 5ed. p. 71

Let $f(x)$ be a function and $\ell \in \mathbb{R}$. We say $f(x)$ *approaches* the limit ℓ (or *converges* to the limit ℓ) as x approaches a if we can make the value of $f(x)$ arbitrarily close to ℓ (as close to ℓ as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

We write

$$\lim_{x \to a} f(x) = \ell.$$

Roughly speaking, $f(x)$ is close to ℓ for all x values sufficiently close to a, with $x \neq a$. The limit "predicts" what should happen at $x = a$ by looking at x values close to but not equal to a.

12.1.1 Precise Definition

Let f be a function defined on some open interval that contains the number a, except possibly a itself. Then we write

$$\lim_{x \to a} f(x) = \ell$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$|f(x) - \ell| < \varepsilon \text{ whenever } 0 < |x - a| < \delta.$$

Detailed discussion on the precise definition of a limit can be found in Stewart, 6ed. p. 88; 5ed. p. 93. This is not directly examinable in MATH1051, but you will need it in later courses.

We can sometimes determine the limit of a function by looking at its graph. Figure 40 shows two functions with the same limit at 5.
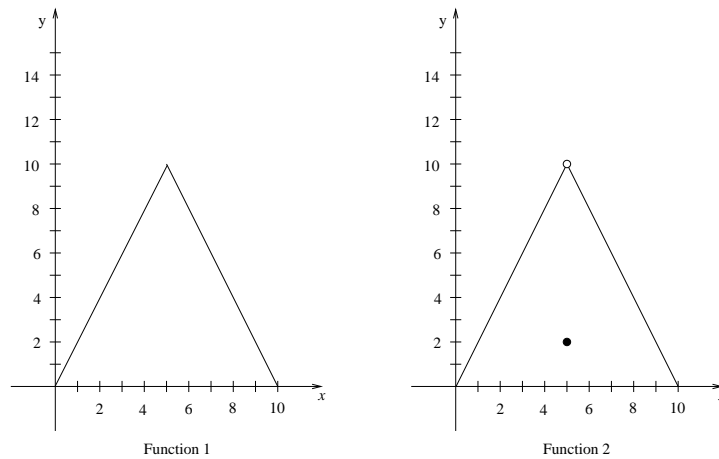
Figure 40: Two functions with limit equal to 10 as $x \to 5$.

Function 1 is described by

$$y = \begin{cases} 2x & \text{for } 0 \le x \le 5, \\ -2x + 20 & \text{for } 5 < x \le 10, \end{cases}$$

while function 2 is described by

$$y = \begin{cases} 2x & \text{for } 0 \le x < 5, \\ 2 & \text{for } x = 5, \\ -2x + 20 & \text{for } 5 < x \le 10. \end{cases}$$

Each function has a limit of 10 as x approaches 5. It does not matter that the value of the second function is 2 when x equals 5 since, when dealing with limits, we are only interested in the behaviour of the function as x approaches 5.

12.1.2   Example: Stewart, 6ed. p. 69; 5ed. p. 74

Investigate $\lim\limits_{x \to 0} \sin \dfrac{\pi}{x}$.

$\sin(\pi/x)$ oscillates rapidly between -1 and 1 as $x \to 0$, so the limit does not exist.

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12.1.3   Properties

Suppose that c is a constant and the limits $\ell = \lim_{x \to a} f(x)$ and $m = \lim_{x \to a} g(x)$ exist for some fixed $a \in \mathbb{R}$. Then

$$
\begin{aligned}
\lim_{x \to a} (f(x) \pm g(x)) &= \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = \ell \pm m \\[2ex]
\lim_{x \to a} cf(x) &= c \cdot \lim_{x \to a} f(x) = c \cdot \ell \\[2ex]
\lim_{x \to a} f(x) \cdot g(x) &= \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = \ell \cdot m \\[2ex]
\lim_{x \to a} \frac{f(x)}{g(x)} &= \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{\ell}{m} \text{ if } m \neq 0
\end{aligned}
$$

12.1.4 Example

Find the value of $\lim\limits_{x \to 1}(x^2 + 1)$.

$$\lim_{x \to 1}(x^2 + 1) = 1 + 1 = 2$$

12.1.5 Example: Stewart, 6ed. p. 67 ex. 1; 5ed. p. 71 ex. 1

Determine the value of $\lim\limits_{x \to 1} \dfrac{x-1}{x^2-1}$.

$$
\begin{aligned}
\lim_{x \to 1} \frac{x-1}{x^2-1} &= \lim_{x \to 1} \frac{x-1}{(x+1)(x-1)} \qquad \text{remember } x \neq 1 \\
&= \lim_{x \to 1} \frac{1}{x+1} = \frac{1}{2}.
\end{aligned}
$$

Note: this means

$$
f(x) = \begin{cases} \dfrac{x-1}{(x+1)(x-1)} & x \neq 1 \\ \dfrac{1}{2} & x = 1. \end{cases}
$$

is a continuous function.

12.1.6   Definition: Infinite Limits

Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a, but not equal to a.
Similarly,

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negatively by taking x sufficiently close to a, but not equal to a.
In these cases, we say that $f(x)$ *diverges* to $\pm\infty$. Note that the limit properties in section 12.1.3 do not necessarily apply if the limits diverge.

12.2   One-sided limits

Consider the piecewise function

$$f(x) = \begin{cases} 1, & x \geq 0 \\ -2, & x < 0. \end{cases}$$

This function has the graph depicted in Figure 41.
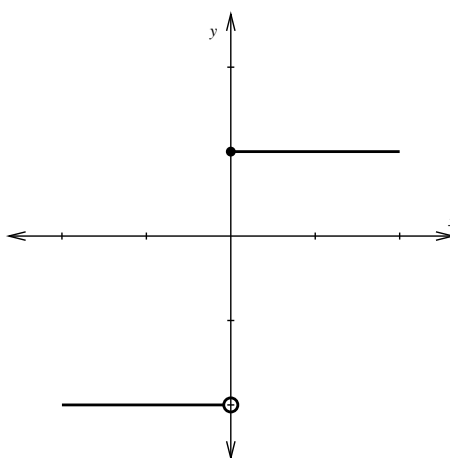
Figure 41: The limit of this function as $x \to 0$ does not exist.

Notice that $\lim_{x \to 0, x > 0} f(x) = 1$, but $\lim_{x \to 0, x < 0} f(x) = -2$. Therefore, the limit as $x \to 0$ does not exist. We can, however, talk about the *one-sided limits*.

In the above example, we say that the limit as $x \to 0$ *from above* (or *from the right*) equals 1 and we write

$$\lim_{x \to 0^+} f(x) = 1.$$

Similarly, we say that the limit as $x \to 0$ *from below* (or *from the left*) equals -2 and we write

$$\lim_{x \to 0^-} f(x) = -2.$$

In general, for $\lim_{x \to a^+} f(x) = \ell$, just consider x with $x > a$ and similarly for $\lim_{x \to a^-} f(x) = \ell$, consider only $x < a$.

12.2.1   Example

Consider both $\lim_{x \to 2^+} \sqrt{x-2}$ and $\lim_{x \to 2^-} \sqrt{x-2}$. Based on your conclusion, does the $\lim_{x \to 2} \sqrt{x-2}$ exist?

$\lim_{x \to 2^+} \sqrt{x-2} = 0$, however $\lim_{x \to 2^-} \sqrt{x-2}$ does not exist.
$\Rightarrow \lim_{x \to 2} \sqrt{x-2}$ does not exist.

12.2.2   Theorem

$\lim_{x \to a} f(x) = \ell$ if and only if

$$\lim_{x \to a^-} f(x) = \ell = \lim_{x \to a^+} f(x).$$

12.2.3   Example

Find $\lim_{x \to 1} f(x)$ where $f(x) = \begin{cases} x^2, & x \geq 1, \\ 2-x, & x < 1. \end{cases}$

$$\begin{aligned}
\lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} x^2 = 1, \\
\lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} (2-x) = 2 - 1 = 1, \\
&\Rightarrow \lim_{x \to 1} f(x) = 1.
\end{aligned}$$

12.3   Theorem: Squeeze principle

Suppose
$$\lim_{x \to a} g(x) = \ell = \lim_{x \to a} h(x)$$
and, for x close to a $(x \neq a)$
$$h(x) \leq f(x) \leq g(x).$$

Then

$$\lim_{x \to a} f(x) = \ell.$$

See the graph in Figure 42.
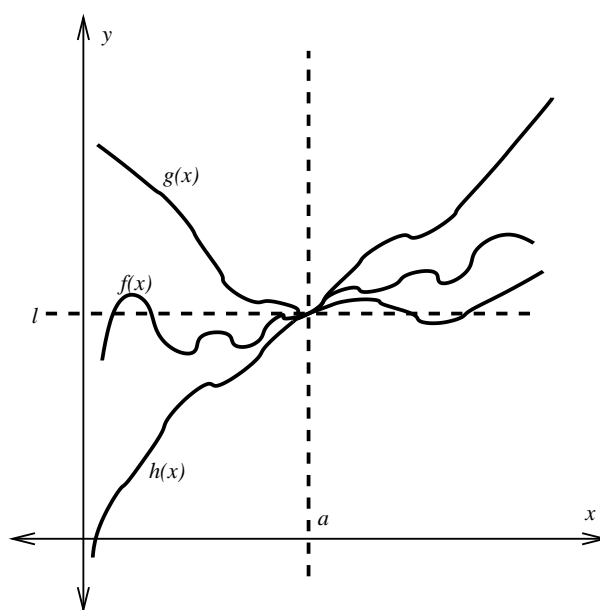
Figure 42: The squeeze principle

12.3.1 Example

Prove that $\lim\limits_{x \to 0} x^2 \sin\left(\dfrac{1}{x}\right) = 0$.

For $x \neq 0$,
$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1.$$

Therefore,
$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2.$$

Take $g(x) = x^2$, and $h(x) = -x^2$ in the squeeze theorem.
As $x \to 0$, $g(x) \to 0$ and $h(x) \to 0$.

So by the squeeze principle $\Rightarrow$

$$\lim_{x \to 0} \ x^2 \sin\left(\frac{1}{x}\right) = 0.$$

12.4 Limits as x approaches infinity

If $f(x)$ approaches ℓ as x gets larger and larger, we write

$$\lim_{x \to \infty} f(x) = \ell.$$

That is, $f(x)$ can be made arbitrarily close to ℓ by taking x sufficiently large. Similarly, we write

$$\lim_{x \to -\infty} f(x) = \ell$$

if $f(x)$ approaches ℓ as x becomes more and more negative.

12.4.1 Example

Find $\lim\limits_{x \to \infty} \frac{1}{x}$.

$$\lim_{x \to \infty} \frac{1}{x} = \lim_{x \to -\infty} \frac{1}{x} = 0.$$

Note: Since
$$\lim_{x \to 0^+} \frac{1}{x} = \infty \text{ and } \lim_{x \to 0^-} \frac{1}{x} = -\infty,$$
then $\lim_{x \to 0} \frac{1}{x}$ doesn't exist. Furthermore, it is irrelevant that $f(x)$ is not actually defined at the point $x = 0$.
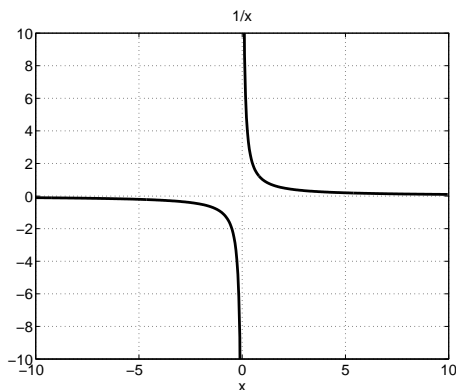
Figure 43: Graph of $f(x) = 1/x$.

12.4.2 Example

Determine $\lim_{x \to \infty} \sin x$, or show that it does not exist.

This limit does not exist since $\sin x$ is a periodic function that oscillates between -1 and 1. (Compare with $\lim_{x \to 0} \sin \frac{\pi}{x}$ in 3.1.4).

12.4.3 Example: ratio of two polynomials, numerator degree = denominator degree

Find $\lim_{x \to \infty} \frac{2x^2 + 3}{3x^2 + x}$.

Put $f(x) = 2x^2 + 3$ and $g(x) = 3x^2 + x$.

Then $\lim_{x \to \infty} f(x) = \infty$ and $\lim_{x \to -\infty} g(x) = \infty$

So the limit has the form $\frac{\infty}{\infty}$.

Technique: we divide top and bottom by the highest power of x in the denominator. The result is two new functions in the numerator and the denominator, the limits of which are both finite.

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$$\lim_{x \to \infty} \frac{2x^2 + 3}{3x^2 + x} = \lim_{x \to \infty} \frac{2 + \frac{3}{x^2}}{3 + \frac{1}{x}}$$

$$= \lim_{x \to \infty} \frac{(2x^2 + 3)/x^2}{(3x^2 + x)/x^2}$$

$$= \lim_{x \to \infty} \frac{2 + \frac{3}{x^2}}{3 + \frac{1}{x}}$$

$$= \frac{2}{3}$$

12.4.4 Example

Find $\displaystyle \lim_{x \to \infty} \frac{8x^3 + 6x^2}{4x^3 - x + 12}$.

In this case, we divide the top and bottom by x^3, and proceed as before.

$$\lim_{x \to \infty} \frac{8x^3 + 6x^2}{4x^3 - x + 12} = \lim_{x \to \infty} \frac{(8x^3 + 6x^2)/x^3}{(4x^3 - x + 12)/x^3}$$

$$= \lim_{x \to \infty} \frac{8 + \frac{6}{x}}{4 - \frac{1}{x^2} + \frac{12}{x^3}}$$

$$= \frac{8}{4}$$

$$= 2$$

12.4.5 Example: ratio of two polynomials, numerator degree > denominator degree

Find $\lim\limits_{x\to\infty}\dfrac{x^2+5}{x+1}$.

> The highest power of x in the denominator is 1, so we have
>
> $$\lim_{x\to\infty}\frac{x^2+5}{x+1}=\lim_{x\to\infty}\frac{x+\frac{5}{x}}{1+\frac{1}{x}}=\infty.$$

12.4.6 Example: ratio of two polynomials, numerator degree < denominator degree

Find $\lim\limits_{x\to\infty}\dfrac{x+1}{x^2+1}$

> $$\lim_{x\to\infty}\frac{x+1}{x^2+1}=\lim_{x\to\infty}\frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{1}{x^2}}=0$$

The general method in the four previous examples has been to:

1. divide the numerator and denominator by the highest power of x in the denominator; and then

2. determine the limits of the numerator and denominator separately.

12.5 Some important limits

The following limits are fundamental. Combined with the properties given in 12.1.3 and the Squeeze Principle given in 12.3, these will enable you to compute a range of other limits.

Here $a \in \mathbb{R}$ is arbitrary.

$$(1) \quad \lim_{x \to a} 1 = 1 \qquad\qquad (2) \quad \lim_{x \to a} x = a \qquad\qquad (3) \quad \lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$(4) \quad \lim_{x \to 0} \frac{1 - \cos(x)}{x^2} = \frac{1}{2} \qquad (5) \quad \lim_{x \to 0} \frac{e^x - 1}{x} = 1.$$

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13   Continuity

13.1   Definition: Continuity

We say that a function f is *continuous at a* if

(i) $f(a)$ is defined (that is, a is in the domain of $f(x)$);

(ii) $\lim_{x \to a} f(x)$ exists; and

(iii) $\lim_{x \to a} f(x) = f(a)$.

If f is not continuous at a, we say that f is *discontinuous at a*, or f has a *discontinuity at a*.
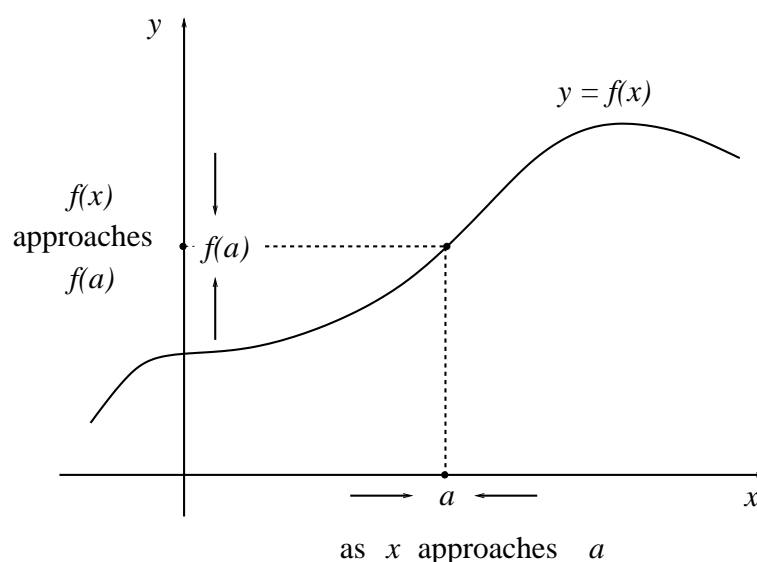
Figure 44: Graphical representation of continuity at $x = a$.

A function may not be continuous at $x = a$ for a number of reasons.

13.1.1   Example

Let

$$f(x) = \begin{cases} x, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

Then $f(x)$ has a discontinuity at $x = 0$. This is because $f(0) = 1$ while $\lim_{x \to 0} f(x) = 0$. Hence Condition (ii) of Definition 13.1 doesn't hold. See Figure 45.
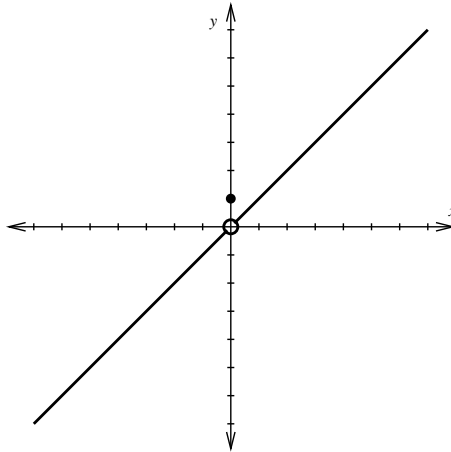
Figure 45: An example of a discontinuous function with discontinuity at $x = 0$.

13.1.2 Example

The function $f(x) = \dfrac{1}{x^2}$ (see Figure 46) is not continuous at $x = 0$, since

$f(0)$ is not defined, i.e. $0 \notin \text{dom}(f)$. Thus Condition (i) in Definition 13.1 doesn't hold.
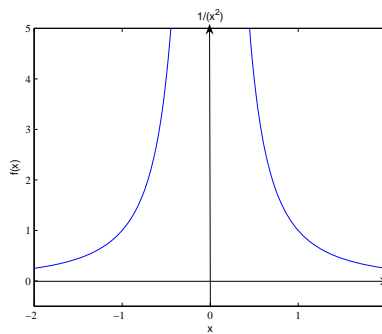
Figure 46: The function $f(x) = 1/x^2$ has a discontinuity at $x = 0$.

13.1.3 Example

$$f(x) = \begin{cases} x + 1, & x \geq 0 \\ x^2, & x < 0 \end{cases}$$

is not continuous at $x = 0$ since

$\lim\limits_{x \to 0} f(x)$ doesn't exist:

$$\begin{aligned} \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} (x + 1) = 1 \\ \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} x^2 = 0 \end{aligned}$$

The limit from above is different to the limit from below.

13.2   Definition: Continuity on intervals

If f is continuous on the *open* interval (a, b), then

f is continuous at c, for all $c \in (a, b)$.

If f is continuous on the *closed* interval $[a, b]$, then f is continuous on (a, b) and

$$\lim_{x \to a^+} f(x) = f(a), \quad \lim_{x \to b^-} f(x) = f(b).$$

The graph of a function that is continuous on an interval can be drawn without lifting the pen.

13.2.1   Examples

- Any polynomial in x is continuous on $\mathbb{R}$. For example, $f(x) = ax^2 + bx + c$ is continuous on $\mathbb{R}$.

- e^x, $|x|$, $\sin x$, $\cos x$ and $\arctan x$ are continuous on $\mathbb{R}$.

- $f(x) = \ln x$ is continuous on $(0, \infty)$.

13.3 Properties

If $f(x)$ and $g(x)$ are continuous at $x = a$ and c is a constant, then

$$(1) \quad f(x) \pm g(x),$$

$$(2) \quad c \cdot f(x),$$

$$(3) \quad f(x) \cdot g(x), \text{ and}$$

$$(4) \quad \frac{f(x)}{g(x)}, \text{ where } g(a) \neq 0,$$

are all continuous at $x = a$.

13.3.1 Theorem: Limit of a composite function - Stewart, 6ed. p. 103 Thm 8; 5ed. p. 107 Thm 8

If f is continuous at b and $\lim\limits_{x \to a} g(x) = b$, then $\lim\limits_{x \to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right).$$

Similarly if $\lim\limits_{x \to \infty} g(x) = b$ and f is continuous at $x = b$, then

$$\lim_{x \to \infty} f(g(x)) = f(b).$$

13.3.2 Corollary: Stewart, 6ed. p. 103 Thm 9; 5ed. p. 108 Thm 9

If g is continuous at a and f is continuous at $g(a)$ then $f \circ g$ is continuous at a.

13.4 Intermediate Value Theorem (IVT) - Stewart, 6ed. p. 104; 5ed. p. 109

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number $c \in (a, b)$ such that $f(c) = N$.
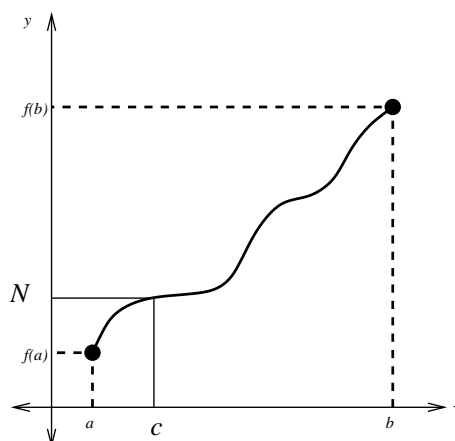
Figure 47: The IVT states that a continuous function takes on *every* intermediate value between the function values $f(a)$ and $f(b)$. That is, we can choose any N between $f(a)$ and $f(b)$ which corresponds to a value c between a and b. Note that this is not necessarily true if f is discontinuous.

13.5 Application of IVT (bisection method)

The *bisection method* is a procedure for approximating the zeros of a continuous function. It first cuts the interval $[a, b]$ in half (say, at a point c), and then decides in which of the smaller intervals ($[a, c]$ or $[c, b]$) the zero lies. This process is repeated until the interval is small enough to give a significant approximation to the zero itself.

We can present this bisection method as an *algorithm*:

(1) Given $[a, b]$ such that $f(a)f(b) < 0$, let $c = \dfrac{a + b}{2}$.

(2) If $f(c) = 0$ then quit; c is a zero of f.

(3) If $f(c) \neq 0$ then:

    (a) If $f(a)f(c) < 0$, replace b by c.

    (b) If $f(a)f(c) > 0$, replace a by c.

(4) If the interval $[a, b]$ is small enough to give a precise enough approximation then quit. Otherwise, go to step (1).
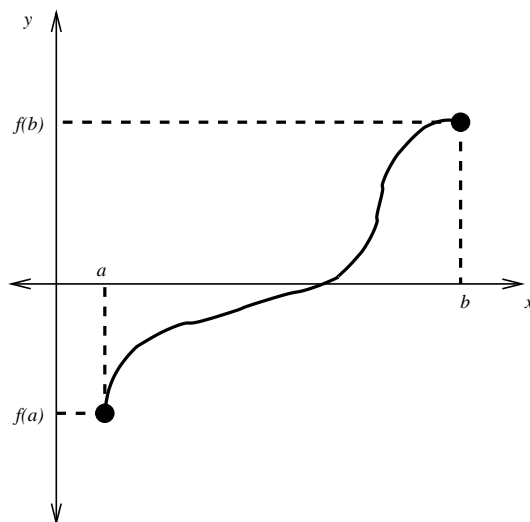
Figure 48: The bisection method is an application of the IVT.

Note that:

$$f(a)f(c) < 0 \quad \Longleftrightarrow \quad f(a) < 0 \text{ and } f(c) > 0$$
$$\text{or} \quad f(a) > 0 \text{ and } f(c) < 0$$
$$f(a)f(c) > 0 \quad \Longleftrightarrow \quad f(a) > 0 \text{ and } f(c) > 0$$
$$\text{or} \quad f(a) < 0 \text{ and } f(c) < 0$$

We will investigate the bisection method in more detail in the lab sessions with MATLAB.

Notes

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14   Derivatives

Finding the instantaneous velocity of a moving object and other problems involving *rates of change* are situations where derivatives can be used as a powerful tool. All rates of change can be interpreted as *slopes* of appropriate tangents. Therefore we shall consider the tangent problem and how it leads to a precise definition of the derivative.

14.1   Tangents
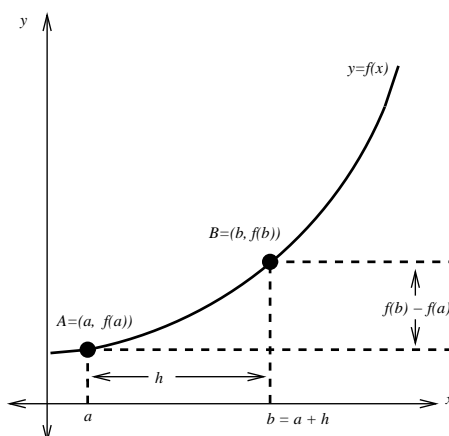
Figure 49: We want to determine the tangent line at the point A.

Consider the graph in Figure 49. We want to determine the tangent line at the point A ($x = a$) on the graph of $y = f(x)$, where $f(x)$ is 'nice enough'. We approximate the tangent at A by the chord AB where B is a point on the curve (close to A) with x-value $a + h$, where h is small. By looking at the graph, we can see that the slope m of the chord AB is given by

$$
\begin{aligned}
m &= \frac{\text{vertical rise}}{\text{horizontal run}} \\
&= \frac{f(b) - f(a)}{b - a} \\
&= \frac{f(a + h) - f(a)}{h}.
\end{aligned}
$$

As the point B gets closer to A, m will become closer to the slope at the point A. To obtain this value, we take the limit as $h \to 0$:

$$
\text{The slope of the tangent to } y = f(x) \text{ at } A = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.
$$

14.2 Definition: Derivative, differentiability

The *derivative* of f at x is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

We say that f is *differentiable* at some point x if this limit exists. Further, we say that f is differentiable on an open interval if it is differentiable at every point in the interval. Note that $f'(a)$ is the slope of the tangent line to the graph of $y = f(x)$ at $x = a$.

We have thus defined a new function f', called the derivative of f. Sometimes we use the Leibniz notation $\dfrac{dy}{dx}$ or $\dfrac{df}{dx}$ in place of $f'(x)$.

Note that if f is differentiable at a, there holds:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

14.2.1 Example

Using the definition of the derivative (i.e. calculating 'by first principles'), show that if $f(x) = x^2 + x$, then $f'(x) = 2x + 1$.

Using the definition for a derivative, we find:

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\[2mm]
&= \lim_{h \to 0} \frac{(x+h)^2 + (x+h) - (x^2 + x)}{h} \\[2mm]
&= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + x + h - x^2 - x}{h} \\[2mm]
&= \lim_{h \to 0} \frac{2xh + h^2 + h}{h} \\[2mm]
&= \lim_{h \to 0} (2x + h + 1) \\[2mm]
&= 2x + 1
\end{aligned}
$$

14.2.2 Example

Show that if $f(x) = e^x$, then $f'(x) = e^x$.

As discussed earlier, we have from the definition of Euler's number e:

$$\lim_{h \to 0} \frac{e^h - 1}{h - 0} = 1, \quad \text{i.e.} \quad \lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

Using the definition of the derivative, we then have

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \\
&= \lim_{h \to 0} \frac{e^x e^h - e^x}{h} \\
&= \lim_{h \to 0} e^x \cdot \frac{e^h - 1}{h} \\
&= e^x \lim_{h \to 0} \frac{e^h - 1}{h} \\
&= e^x
\end{aligned}
$$

14.2.3 Some useful derivatives

Other useful derivatives are

$$
\begin{aligned}
\frac{d}{dx}(\text{constant}) &= 0 \\
\frac{d}{dx}(x) &= 1 \\
\frac{d}{dx}(x^\alpha) &= \alpha x^{\alpha - 1}, \text{ for } \alpha \in \mathbb{R}, \ x \geq 0 \text{ or for } x \in \mathbb{R} \text{ if } \alpha \in \mathbb{R}^+ \\
\frac{d}{dx}(\sin x) &= \cos x \\
\frac{d}{dx}(\cos x) &= -\sin x
\end{aligned}
$$

14.3 Theorem: Differentiability implies continuity

If f is differentiable at a, then

> f is continuous at a.

The converse, however, is false. That is, if f is continuous at a, then it is not necessarily differentiable. (See the following example.)

14.3.1 Example

The function $f(x) = |x|$ is continuous, but not differentiable in $x = 0$, since

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h - 0}{h} = 1$$

and

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{-h - 0}{h} = -1 \neq 1,$$

meaning that $\lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$ doesn't exist.

14.4 Rates of change - Stewart, 6ed. p. 117; 5ed. p. 117

Recall that the slope of the tangent line to $y = f(x)$ at $x = a$ measures the *rate of change* of y with respect to x at the point $x = a$.

14.4.1 Example: Velocity

Velocity v is the rate at which displacement s changes with respect to time t, so

$$v(t) = \frac{ds}{dt} = s'(t) = \dot{s}(t).$$

Similarly *acceleration a* is the rate at which velocity v changes with respect to time t, so

$$a(t) = \frac{dv}{dt} = v'(t) = \dot{v}(t).$$

Note: s, a, v are **vector** quantities: but in one dimension, this just means signed ($+$ or $-$).

14.4.2 Example (blood flow in an artery) - Stewart, 6ed. p. 176 ex. 7; 5ed. p. 163 ex. 7

Blood flowing through a blood vessel can be described by the law of laminar flow. We model the artery or vein as a cylinder. The closer the blood is to the artery wall, the slower it will flow due to friction. Conversely, the velocity of the blood is greatest at the centre of the blood vessel (cylinder). In mathematical terms, this law can be stated

$$v = \frac{P}{4\eta l}(R^2 - r^2),$$

where η is the viscosity of the blood, P is the pressure difference between the ends of the artery, l is the length of the blood vessel, R is the radius of the vessel and r is the distance from the centre of the vessel (the only variable in this case); see Figure 50.

We now seek to find a general expression for the *velocity gradient* of the blood, that is, an expression for the rate of change of velocity with respect to r:
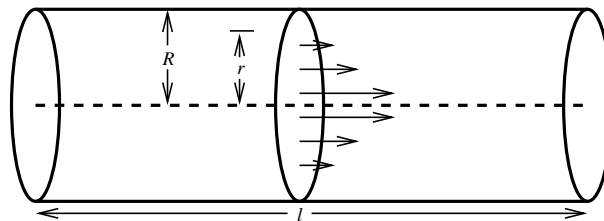
Figure 50: The closer the blood is to the vein wall, the smaller its velocity.

$$
\begin{aligned}
\text{velocity gradient} \quad &= \quad \frac{dv}{dr} \\
&= \quad \frac{d}{dr}\left(\frac{PR^2}{4\eta l} - \frac{Pr^2}{4\eta l}\right) \\
&= \quad 0 - \frac{d}{dr}\left(\frac{Pr^2}{4\eta l}\right) \\
&= \quad -\frac{Pr}{2\eta l}.
\end{aligned}
$$

This tells us at what rate the velocity of the blood decreases the further it is from the centre of the blood vessel.

These types of problems are studied in later courses on fluid mechanics and other related topics.

14.5 Higher derivatives

Differentiating a function $y = f(x)$ n times, if possible, gives the nth derivative of f, usually denoted $f^{(n)}(x)$, $\dfrac{d^n f}{dx^n}$ or $\dfrac{d^n y}{dx^n}$.

Therefore acceleration can be represented as

$$
\begin{aligned}
a(t) &= \frac{dv}{dt} \\
&= \frac{d}{dt}\left(\frac{ds}{dt}\right) \\
&= \frac{d^2 s}{dt^2} \\
&\equiv s''(t) \\
&\equiv s^{(2)}(t) \\
&\equiv \ddot{s}(t).
\end{aligned}
$$

Sometimes when we deal with time derivatives we write $v(t) = \dot{s}(t)$ or $a(t) = \ddot{s}(t)$ and so on.

14.6 Rules for differentiation

Suppose that $f(x)$ and $g(x)$ are differentiable functions, and let c be a constant. Then we have the following rules:

(i) $(cf)' = cf'$;

(ii) $(f \pm g)' = f' \pm g'$;

(iii) $(fg)' = f'g + fg'$ (the product rule); and

(iv) $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$, $g \neq 0$ (the quotient rule).

In the Leibniz notation, if $f(x)$ and $g(x)$ are both differentiable functions, and c is a constant, then the rules for differentiation can be expressed as:

(i) $\quad \dfrac{d}{dx}(cf) = c\dfrac{df}{dx}$;

(ii) $\quad \dfrac{d}{dx}(f \pm g) = \dfrac{df}{dx} \pm \dfrac{dg}{dx}$;

(iii) $\quad \dfrac{d}{dx}(fg) = \dfrac{df}{dx}g + f\dfrac{dg}{dx}$ (the product rule); and

(iv) $\quad \dfrac{d}{dx}\left(\dfrac{f}{g}\right) = \dfrac{\frac{df}{dx}g - f\frac{dg}{dx}}{g^2}$, for $g \neq 0$ (the quotient rule).

14.6.1 Example

Use the rules above to find the derivative of $f(x) = \tan x = \dfrac{\sin x}{\cos x}$.

For $x \neq (n + \frac{1}{2})\pi$, $n \in \mathbb{Z}$, there holds $\cos(x) \neq 0$, so use the quotient rule:

$$
\begin{aligned}
f'(x) &= \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \quad (\text{ since } \cos^2 x + \sin^2 x = 1) \\
&= \sec^2 x.
\end{aligned}
$$

14.7 The chain rule - Stewart, 6ed. p. 156; 5ed. p. 176

If g and h are both differentiable and $f = g \circ h$ is the composite function defined by $f(x) = g(h(x))$, then f is differentiable and f' is given by the product

$$f'(x) = g'(h(x))h'(x).$$

In the Leibniz notation, if $y = g(u)$ and $u = h(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

14.7.1 Example

Differentiate $y = \sqrt{\sin x}$.

Let $u = \sin x$. Then $y = u^{1/2}$. Thus, we have

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\
&= \left(\frac{1}{2} u^{-1/2} \right)(\cos x) \\
&= \frac{\cos x}{2\sqrt{u}} \\
&= \frac{\cos x}{2\sqrt{\sin x}}
\end{aligned}
$$

14.7.2   Example: Rocket tracking application

A rocket is launched vertically and tracked by an observation station 5km from the launch pad. The angle of elevation θ is observed to be increasing at $3°/$s when $\theta = 60°$. What is the velocity of the rocket at that instant?

Note that
$$\theta = \frac{\pi}{3} \text{ (radians)}$$
and for $\theta = \frac{\pi}{3}$:

$$\frac{d\theta}{dt} = 3°/\text{s}$$
$$= \frac{\pi}{60} \text{ radians/s}$$

We have $y = 5\tan\theta$, so $\dfrac{dy}{dt} = 5\sec^2\theta$.
By the chain rule:

$$\frac{dy}{dt} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dt}$$
$$= (5\sec^2\frac{\pi}{3})\left(\frac{\pi}{60}\right)$$
$$= \frac{\pi}{3} \text{ km/s.}$$

14.8 Derivative of inverse function

Suppose $y = f^{-1}(x)$, where f^{-1} is the inverse of f. To obtain $\dfrac{dy}{dx}$ we use

$$x = f(f^{-1}(x)) = f(y).$$

Differentiating both sides with respect to x using the chain rule gives

$$
\begin{aligned}
1 &= \frac{d}{dx}(x) \\
&= \frac{d}{dx} f(y) \\
&= \frac{df}{dy} \cdot \frac{dy}{dx} \\
&= \frac{dx}{dy} \cdot \frac{dy}{dx}
\end{aligned}
$$

Hence, if $\dfrac{dx}{dy} \neq 0$:

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}.$$

14.8.1 Example

Find the derivative of $y = \ln x$.

$y = \ln x \Rightarrow x = e^y$. Now (noting that e^y is never 0)

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{1}{\left(\frac{dx}{dy}\right)} \\
&= \frac{1}{e^y} \\
&= \frac{1}{x}. \\
\Rightarrow \frac{d(\ln x)}{dx} &= \frac{1}{x}.
\end{aligned}
$$

Important: Our goal is to find an explicit expression in terms of x.

14.8.2 Example

Find the derivative of $y = \arcsin x$

First note that the domain for the arcsin function is $(-1, 1)$ and the range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

We have $x = \sin y$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{1}{\left(\frac{dx}{dy}\right)} \\
&= \frac{1}{\cos y}, \left(\text{note } \cos(y) > 0 \text{ for } y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right) \\
&= \frac{1}{\sqrt{1 - \sin^2 y}} \\
&= \frac{1}{\sqrt{1 - x^2}}.
\end{aligned}
$$

Therefore,
$$
\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1 - x^2}}, \quad x \in (-1, 1).
$$

14.8.3 Other useful derivatives

$$
\begin{aligned}
\frac{d}{dx}(\arccos x) &= -\frac{1}{\sqrt{1 - x^2}} \quad \text{for } x \in (-1, 1) \\
\frac{d}{dx}(\arctan x) &= \frac{1}{1 + x^2} \quad \text{for } x \in \mathbb{R}
\end{aligned}
$$

14.9   L'Hôpital's rule - Stewart, 6ed. p. 472; 5ed. p. 494

Suppose that f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \text{ and } \lim_{x \to a} g(x) = 0$$

or

$$\lim_{x \to a} f(x) = \pm\infty \text{ and } \lim_{x \to a} g(x) = \pm\infty.$$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right exists.

14.9.1   Example

Verify $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$.

Note that
$$\lim_{x \to 0} \sin x = \lim_{x \to 0} x = 0.$$

Hence, we *cannot* use
$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to 0} f(x)}{\lim\limits_{x \to 0} g(x)}.$$

Instead, we use L'Hôpital's rule to get:

$$\lim_{x \to 0} \frac{\sin x}{x} \;=\; \lim_{x \to 0} \frac{\cos x}{1}, \text{ if this limit exists,}$$

$$=\; 1.$$

14.9.2   Example

Find $\lim\limits_{x\to 0^+} x \ln x$.

Rewrite $x \ln x$ as $\dfrac{\ln x}{1/x}$ and note that in this case

$$\lim_{x\to 0^+} \ln x = -\infty \quad \text{and} \quad \lim_{x\to 0^+} \frac{1}{x} = \infty.$$

Hence we use L'Hôpital's rule to give:

$$\begin{aligned}
\lim_{x\to 0^+} \frac{\ln x}{1/x} &= \lim_{x\to 0^+} \frac{1/x}{-1/x^2} \\
&= \lim_{x\to 0^+} (-x) \\
&= 0.
\end{aligned}$$

14.9.3   Example

Verify that $\lim\limits_{h\to 0} \frac{e^h - 1}{h} = 1$.

Begin by noting that
$$\lim_{h\to 0}(e^h - 1) = 0.$$

Hence,

$$\begin{aligned}
\lim_{h\to 0} \frac{e^h - 1}{h} &= \lim_{h\to 0} \frac{e^h}{1} \\
&= \frac{1}{1} \\
&= 1.
\end{aligned}$$

14.10   The Mean Value Theorem (MVT)

Let f be continuous on $[a, b]$ and differentiable on (a, b). Then

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

for some c, where $a < c < b$.

Note $f'(c)$ is the slope of $y = f(x)$ at $x = c$ and $\frac{f(b)-f(a)}{b-a}$ is the slope of the chord joining $A = (a, f(a))$ to $B = (b, f(b))$. (Stewart, 6ed. p. 216; 5ed. p. 236)

14.11   Definition: Increasing/Decreasing

A function f is called *strictly increasing* on an interval I if

$f(x_1) < f(x_2)$ whenever $x_1 < x_2$,

while f is called *strictly decreasing* on I if

$f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

This leads to the following test.

14.12   Increasing/Decreasing test - Stewart, 6ed. p. 221; 5ed. p. 240

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

(a) If $f'(x) > 0$ on (a, b), then f is strictly increasing on $[a, b]$.

(b) If $f'(x) < 0$ on (a, b), then f is strictly decreasing on $[a, b]$.

(c) If $f'(x) = 0$ on (a, b), then f is constant on $[a, b]$.

Proof of (a):
Let $a \leq d < e \leq b$. It suffices to show $f(e) > f(d)$. Applying the MVT to f on $[d, e]$, there exists $c \in (d, e)$ with

$$
\begin{aligned}
\frac{f(e) - f(d)}{e - d} &= f'(c) \\
&> 0. \\
\Rightarrow f(e) - f(d) &> 0 \\
\text{i.e.} \quad f(e) &> f(d).
\end{aligned}
$$

The proofs of (b) and (c) are similar.

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14.12.1 Example

Test to see on what intervals the function $f(x) = x^3 + x$ is increasing or decreasing.

$$
\begin{aligned}
&\text{We have } f(x) = x^3 + x \\
&\Rightarrow f'(x) = 3x^2 + 1 > 0, \quad \text{for all } x \in \mathbb{R} \\
&\quad \therefore \quad f(x) \text{ is strictly increasing on } \mathbb{R}.
\end{aligned}
$$

14.13 Definition: Local maxima and minima

A function f has a *local maximum* at a if

$f(a) \geq f(x)$ for all x in some open interval containing a (i.e. for x near a).

Similarly, f has a local minimum at b if

$f(b) \leq f(x)$ for all x in some open interval containing b.

14.13.1 Definition (critical point)

A function f is said to have a *critical point* at $x = a$, $a \in \mathrm{dom}(f)$ if

$f'(a) = 0$ or if $f'(a)$ does not exist.

14.13.2 Example
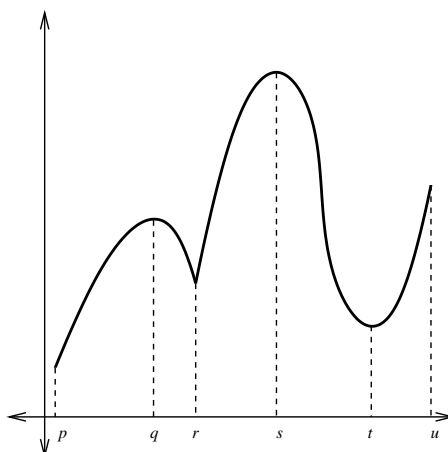
Figure 51: What do the points $x = p, q, r, s, t, u$ represent?

Consider the function in Figure 51. The domain of this function is the interval $[p, u]$. Which points represent the critical points? Which points represent the global maximum and minimum? Which points represent local maxima and minima?

> The global maximum is at point s. The global minimum is at point p. Local maxima are at points q and s. Local minima are at points r and t.

14.13.3 Definition: Global maximum and minimum

Let f be a function defined on the interval $[a, b]$, and $c \in [a, b]$.
Then f has a *global maximum* at c if

> $f(c) \geq f(x)$ for all $x \in [a, b]$.

Similarly f has a *global minimum* at c if

> $f(c) \leq f(x)$ for all $x \in [a, b]$.

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14.13.4   Theorem

If f has a local maximum/minimum at $x = c$ and $f'(c)$ exists then $f'(c) = 0$.

Proof (for max):

For $x < c$, x near c:

$$f(x) - f(c) \leq 0$$
$$\Rightarrow \frac{f(x) - f(c)}{x - c} \geq 0$$
$$\Rightarrow f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0$$

Similarly for $x > c$, x near c:

$$f(x) - f(c) \geq 0$$
$$\Rightarrow \frac{f(x) - f(c)}{x - c} \leq 0$$
$$\Rightarrow f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \leq 0$$

Thus $f'(c) = 0$.

14.14   First derivative test

Let f be continuous on $[a, b]$ and differentiable on (a, b), and let $c \in (a, b)$.

(a) If $f'(x) > 0$ for $a < x < c$ and $f'(x) < 0$ for $c < x < b$

then f has a local maximum at c.

(b) If $f'(x) < 0$ for $a < x < c$ and $f'(x) > 0$ for $c < x < b$

then f has a local minimum at c.

Proof:

(a) Since $f'(x) > 0$ for $a < x < c$ and f continuous on $(a, c]$, f is strictly increasing on $(a, c]$ by 14.12(a). Thus $f(c) > f(x)$ for $x < c$.

Similarly using 14.12(b), $f(c) > f(x)$ for $c < x < b$.

Thus $f(c) \geq f(x)$ for all $x \in \mathbb{R}$ and f has a maximum at $x = c$.

(b)Analogous.

14.15 Second derivative test

Suppose f'' exists and is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then

f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then

f has a local maximum at c.

(c) If $f'(c) = 0$ and $f''(c) = 0$, then the test fails and we get no useful information. In this case the first derivative test should be used.

14.15.1 Example

Find local minima and maxima of $f(x) = x^3$ on $\mathbb{R}$.

second derivative test:

$$
\begin{aligned}
f'(x) &= 3x^2 \text{ and } f''(x) = 6x. \\
f'(x) &= 0 \Rightarrow 3x^2 = 0 \Rightarrow x = 0. \\
f''(0) &= 0 \Rightarrow \text{ no information, use first derivative test.}
\end{aligned}
$$

first derivative test:

$$
\begin{aligned}
f'(x) &= 3x^2. \\
\underline{x < 0}: \quad & 3x^2 > 0. \\
\underline{x > 0}: \quad & 3x^2 > 0. \\
& \Rightarrow f(x) \text{ is always increasing,} \\
& \quad \text{therefore there is no extremal value at } x = 0. \\
& \Rightarrow \text{ no local extrema (0 is a } \textit{point of inflection}).
\end{aligned}
$$

14.15.2 Example

Find local extrema of $f(x) = x^4$.

---

second derivative test:

$$
\begin{aligned}
f'(x) &= 4x^3 \text{ and } f''(x) = 12x^2. \\
f'(x) &= 0 \Rightarrow 4x^3 = 0 \Rightarrow x = 0. \\
f''(0) &= 0 \Rightarrow \text{ no information, use first derivative test.}
\end{aligned}
$$

first derivative test:

$$
\begin{aligned}
f'(x) &= 4x^3. \\
\underline{x < 0}: &\quad 4x^3 < 0. \\
\underline{x > 0}: &\quad 4x^3 > 0. \\
&\Rightarrow \text{ change of sign at } x = 0, \text{ from negative to positive.} \\
&\Rightarrow \text{ local minimum at } x = 0, \text{ with value } f(0) = 0.
\end{aligned}
$$

14.15.3 Example: Beehive cell - Stewart, 6ed. p. 264 qn 43; 5ed. p. 285 qn 39

In a beehive, each cell is a regular hexagonal prism, open at one end with a trihedral angle at the other end (see Figure 52). It is believed that bees form their cells in such a way as to minimise the surface area for a given volume, thus using the least amount of wax in cell construction. Examination of these cells has shown that the measure of the apex angle θ is amazingly consistent. Based on the geometry of the cell, it can be shown that the surface area S is given by

$$S(\theta) = 6sh + \frac{3s^2}{2\sin\theta}(\sqrt{3} - \cos\theta)$$

where s, the length of the sides of the hexagon, and h, the height, are constants.
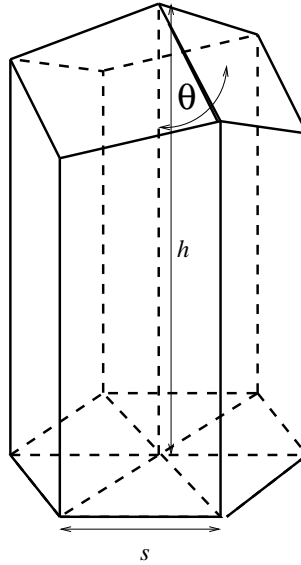
Figure 52: The beehive cell. The open front is pictured on the bottom and the closed end with the trihedral angle on top.

(a) Calculate $\dfrac{dS}{d\theta}$

(b) What angle should the bees prefer?

Actual measurements of the angle θ in beehives have been made, and the measures of these angles seldom differ from the calculated value by more than $2°$.

$$
\begin{aligned}
S(\theta) &= 6sh + \frac{3s^2}{2\sin\theta}(\sqrt{3} - \cos\theta) \quad s, h \text{ constant.}\\
&= 6sh + \frac{3s^2}{2} \cdot \frac{\sqrt{3} - \cos\theta}{\sin\theta}
\end{aligned}
$$

Then:

$$
\begin{aligned}
\frac{dS}{d\theta} &= \frac{3s^2}{2} \cdot \frac{\sin\theta\sin\theta - (\sqrt{3} - \cos\theta)\cos\theta}{\sin^2\theta}\\
&= \frac{3s^2}{2} \cdot \frac{\sin^2\theta + \cos^2\theta - \sqrt{3}\cos\theta}{\sin^2\theta}\\
&= \frac{3s^2}{2\sin^2\theta}(1 - \sqrt{3}\cos\theta).
\end{aligned}
$$

$$\frac{dS}{d\theta} = 0 \quad \Rightarrow \quad \frac{3s^2}{2\sin^2\theta}(1 - \sqrt{3}\cos\theta) = 0$$

$$\Rightarrow \quad 1 - \sqrt{3}\cos\theta = 0$$

$$\Rightarrow \quad \cos\theta = \frac{1}{\sqrt{3}} \quad (\Rightarrow \sin\theta = \sqrt{2/3}).$$

Note that the angle θ should satisfy $0 \le \theta \le \frac{\pi}{2}$ (from a geometry argument). Therefore $\theta = \arccos(\frac{1}{\sqrt{3}}) \approx 55°$ gives the critical point.

Using the second derivative test

$$\frac{d^2S}{d\theta^2} = \frac{3s^2}{2\sin^3\theta}(\sqrt{3} + \sqrt{3}\cos^2\theta - 2\cos\theta) > 0 \text{ when } \cos\theta = \frac{1}{\sqrt{3}}.$$

Therefore we have a local minimum at $\theta \simeq 55°$. (This is in fact a global minimum too.)

14.16 Extreme value theorem - Stewart, 6ed. p. 206; 5ed. p. 225

If f is continuous on a closed interval $[a, b]$, then

---

f attains its global maximum and a global minimum value on $[a, b]$. These values occur either at an end point or an interior critical point.

Calculate the value of f at each of these points and then work out the largest value (global maximum) and the smallest value (global minimum).

Bonus theorem: a) If f is differentiable on (a, b) and only has one critical point, which is a local min; then the global minimum of f occurs at this point.
b) If f is differentiable on (a, b) and only has one critical point, which is a local max; then the global maximum of f occurs at this point.

14.16.1 Example: Stewart, 6ed. p. 209 ex. 8; 5ed. p. 228 ex. 8

Find the global maximum and minimum of the function $f(x) = x^3 - 3x^2 + 1$ on the interval $[-\frac{1}{2}, 4]$.

Note that
$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$
is defined everywhere and $f'(x) = 0$ at $x = 0$ or 2. Both critical points therefore lie in $(-\frac{1}{2}, 4)$.

Therefore the global maximum and minimum are either at the end points $(x = -\frac{1}{2}$ or 4) or at the interior critical points $(x = 0$ or 2):

$$\begin{aligned} f(0) &= 1 \\ f(2) &= -3 \\ f\left(-\frac{1}{2}\right) &= \frac{1}{8} \\ f(4) &= 17 \end{aligned}$$

Therefore the global maximum is 17 at $x = 4$ and the global minimum is -3 at $x = 2$.

14.17 Optimization problems

14.17.1 Example: Optimal rowing

A rowing team launches a boat from point A on a bank of a straight river 3km wide, and wants to reach point B, 8km downstream on the opposite bank as quickly as possible; see Figure 53. They could row their boat directly across the river to a point C and then run to B, or they could row directly to B, or they could row to some point D between C and B and then run to B. If they can row 6km/h and run 8km/h, where should they land to reach B as soon as possible? (Assume that the speed of the water is negligible compared to the speed at which they row.)
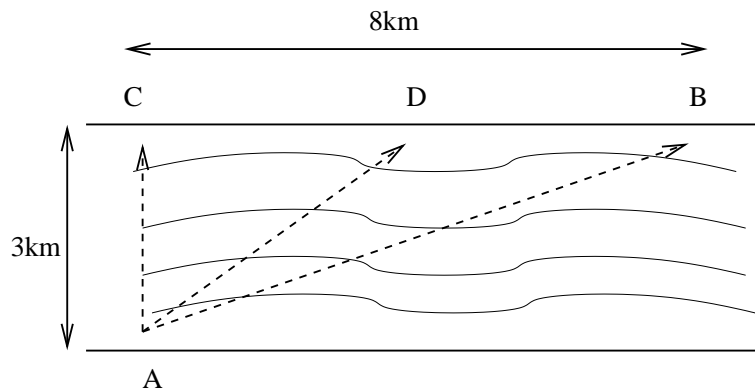
Figure 53: How should the team travel from A to B in the shortest possible time?

Let $x = |CD|$ so $|AD| = \sqrt{9 + x^2}$ and $|DB| = 8 - x$ (km).

Aim: minimize the time taken. We need $|AD|$ and $|DB| \geq 0$, so $0 \leq x \leq 8$.

$$
\begin{aligned}
t(x) &= \frac{\text{distance}}{\text{speed}} \\
&= \frac{1}{6}|AD| + \frac{1}{8}|DB| \\
&= \frac{1}{6}\sqrt{9 + x^2} + \frac{1}{8}(8 - x), \quad 0 \leq x \leq 8.
\end{aligned}
$$

$$
\text{So } t'(x) = \frac{x}{6\sqrt{9 + x^2}} - \frac{1}{8}.
$$

Critical points: $t'(x) = 0$

$$\Rightarrow \quad \frac{x}{6\sqrt{9+x^2}} = \frac{1}{8}$$
$$\Rightarrow \quad 8x = 6\sqrt{9+x^2}$$
$$\Rightarrow \quad 64x^2 = 36(9+x^2)$$
$$\Rightarrow \quad 28x^2 = 324$$
$$\Rightarrow \quad x = \pm\frac{9}{\sqrt{7}}.$$

Hence $x = \frac{9}{\sqrt{7}}$ is the only critical point in (0,8).

Candidates for the global minimum are $0, \frac{9}{\sqrt{7}}, 8$
Calculate:

$$
\begin{aligned}
t(0) &= 1.5; \\
t\left(\frac{9}{\sqrt{7}}\right) &= 1 + \frac{\sqrt{7}}{8} \approx 1.33; \\
t(8) &= \frac{\sqrt{73}}{6} \approx 1.42.
\end{aligned}
$$

Therefore the global minimum is 1.33 hours at $x = \frac{9}{\sqrt{7}}$ km.

14.17.2 Example: Optimal farming

Some farmers have a certain amount of fencing to build a rectangular enclosure which is to be subdivided into 3 parts by 2 fences parallel to one of the sides. How should they do it so the area enclosed is maximised?

Let x and y be the lengths of the sides. Then the total length of fencing, ℓ, is

$$\ell = 4x + 2y \quad \circledast$$

$$\Rightarrow y = \frac{1}{2}\ell - 2x$$

Since $0 \leq x, y$ we have $0 \leq x \leq \frac{\ell}{4}$.

We want to maximize the area, $A = xy$, subject to the constraint $\circledast$.

Hence we want to find the global maximum value of the area

$$A(x) = xy = x\left(\frac{\ell}{2} - 2x\right)$$

for $x \in [0, \ell/4]$.

Critical points: $A'(x) = 0$

$$\Rightarrow \quad \frac{\ell}{2} - 4x = 0$$

$$\Rightarrow \quad \frac{\ell}{2} = 4x$$

$$\Rightarrow \quad x = \frac{\ell}{8}.$$

Candidates: $x = 0, \ell/8$ or $\ell/4$.

$$
\begin{aligned}
A(0) &= 0; \\
A\left(\frac{\ell}{8}\right) &= \frac{\ell}{8}\left(\frac{\ell}{2} - \frac{2\ell}{8}\right) = \frac{\ell^2}{32}; \\
A\left(\frac{\ell}{4}\right) &= 0.
\end{aligned}
$$

So, the maximum value of area occurs when $x = \frac{\ell}{8}$ and $y = \frac{\ell}{4}$.

Notes

Notes

Notes

15 Integration

15.1 Definition: Anti-derivative - Stewart, 6ed. p. 274; 5ed. p. 300

A function F is called an *anti-derivative* of f or, a primitive of f on an interval I if

$$F'(x) = f(x), \quad \text{for all } x \in I.$$

See the table below for some examples of anti-derivatives. We use the notation $F' = f$, $G' = g$ and c is a constant.

Function	Antiderivative
$cf(x)$	$cF(x)$
$f(x) + g(x)$	$F(x) + G(x)$
$x^\alpha, (\alpha \neq -1)$	$\dfrac{x^{\alpha+1}}{\alpha+1}$
$\sin x$	$-\cos x$

Function	Antiderivative
$\cos x$	$\sin x$
$\sec^2 x$	$\tan x$
$\dfrac{1}{x}$	$\ln x$
e^x	e^x

The most general anti-derivative can be obtained from those given in the table by adding a constant. Notice that if $\dfrac{d}{dx}F(x) = f(x)$ then $\dfrac{d}{dx}(F(x) + C) = f(x)$ is also true where C is any constant (independent of x).

15.1.1 Example

Find the most general anti-derivative of the function $f(x) = x^2 + 3x$.

Using the table, we have

$$F(x) = \frac{1}{3}x^3 + \frac{3}{2}x^2 + C,$$

where C is a constant.

15.1.2 Example

If $f''(x) = x - \sqrt{x}$, find $f(x)$.

If $f''(x) = x - \sqrt{x} = x - x^{1/2}$, then

$$f'(x) = \frac{x^2}{2} - \frac{2x^{3/2}}{3} + C$$

and therefore

$$f(x) = \frac{x^3}{6} - \frac{4x^{5/2}}{15} + Cx + D$$

where C and D are constants.

15.2 Definition: Indefinite integral

The *indefinite integral* $\int f(x)dx$, of a suitable function $f(x)$, is defined by

$$\int f(x)dx = F(x) + c,$$

where $F(x)$ is any anti-derivative of $f(x)$ and c is an arbitrary constant called the *constant of integration*. Thus $\int f(x)dx$ gives all the anti-derivatives of $f(x)$.

15.3 Area under a curve - Stewart, 6ed. pp. 289-296; 5ed. p. 315-322

Consider the problem of finding the area under a curve given by the graph of a continuous, non-negative function $y = f(x)$ between two points $x = a$ and $x = b$; see Figure 54.
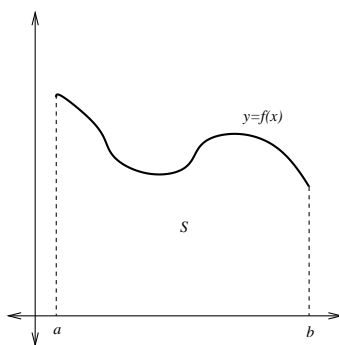
Figure 54: We need to define the area of the region $S = \{(x, y)|a \le x \le b, 0 \le y \le f(x)\}$.

Although we might have an intuitive notion of what we mean by area, how do we *define* the area under such a curve?

We start by subdividing S into n strips $S_1, S_2, \ldots, S_n$ of equal width. The width of the interval $[a, b]$ is $b - a$ so the width of each strip is

$$\Delta x = \frac{b - a}{n}.$$

These strips divide the interval $[a, b]$ into n subintervals

$$[x_0, x_1], \ [x_1, x_2], \ \ldots, \ [x_{n-1}, x_n]$$

where $x_0 = a$ and $x_n = b$. If c_i is *any* point in the ith subinterval $[x_{i-1}, x_i]$, then we can approximate the ith strip S_i by a rectangle of width Δx and height $f(c_i)$, which is the value of f at the point c_i (see Figure 55).
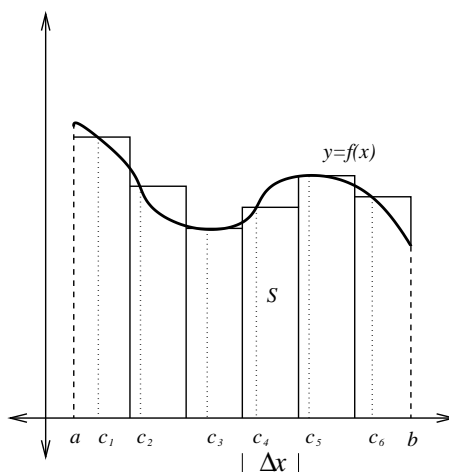
Figure 55: A method of approximating the area of the region S.

The area of such a rectangle is $f(c_i)\Delta x$. Therefore we can approximate the area of S by taking the sum of the area of these rectangles. We call this sum R_n:

$$\begin{aligned} R_n &= f(c_1)\Delta x + f(c_2)\Delta x + \ldots + f(c_n)\Delta x \\ &= \sum_{i=1}^{n} f(c_i)\Delta x \end{aligned}$$

Now what happens if we increase n? Consider the graph in Figure 56. It seems that as Δx becomes smaller (when n becomes larger), the approximation gets better.

Therefore we *define* the area A of the region S as follows.
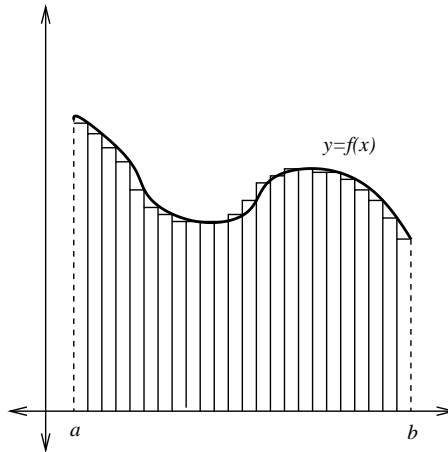
Figure 56: The smaller the width of the rectangle, the better the approximation, in general. Notice that in this diagram we chose $c_i = a + i\Delta x$.

15.3.1   Definition: Area of a region, Riemann sum

The area A of the region S that lies under the graph of the continuous, non-negative function f is

$$
\begin{aligned}
A &= \lim_{n \to \infty} R_n \\
&= \lim_{n \to \infty} [f(c_1)\Delta x + f(c_2)\Delta x + \ldots + f(c_n)\Delta x] \\
&= \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i)\Delta x.
\end{aligned}
$$

This limit always exists and is independent of the choice of c_i. This can be proved using the fact that f is continuous on $[a, b]$. This sum R_n is called a *Riemann sum*.

Note that $\lim_{n \to \infty} [f(c_1)\Delta x + \cdots + f(c_n)\Delta x]$ is called the *Riemann integral* of f on $[a, b]$ and is denoted $\int\limits_{a}^{b} f(x)dx$.

IMPORTANT: this definition can be carried over to functions which are not necessarily non-negative. The intuitive interpretation of A as an area is then no longer valid. The quantity A should then be viewed as a "signed area", i.e.

$A =$ area above x-axis, below graph $-$ area below x-axis, above graph.

Stewart, 6ed. p. 301; 5ed. p. 328

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15.4 Fundamental Theorem of Calculus

15.4.1 Theorem: Fundamental Theorem of Calculus

(a) If f is continuous on $[a, b]$, $a \le x \le b$, then

$$A(x) = \int_a^x f(t)dt$$

gives an antiderivative of $f(x)$ such that $A(a) = 0$.

(b) If $F(x)$ is any antiderivative of $f(x)$, then

$$F(b) - F(a) = \int_a^b f(t)dt.$$

15.4.2 Proof (outline)

(a) Consider $f \ge 0$. Let $A(x) = \int_a^x f(t)dt$, that is let $A(x)$ be the function that describes the area below the graph of the function f above the interval $[a, x]$; thus $A(a) = 0$. For $h > 0$ small, consider

$$A(x + h) - A(x) \quad = \quad \text{area above } [a, x + h] - \text{ area above } [a, x]$$

$$\begin{aligned} &= \quad \text{area above } [x, x + h] \text{ and below } f \\ &\approx \quad h \cdot f(x) \end{aligned}$$

$$\Rightarrow \quad \frac{A(x + h) - A(x)}{h} \approx f(x)$$

Taking the limit $h \to 0$ then yields (this requires justification)

$$A'(x) = \lim_{h \to 0} \frac{A(x + h) - A(x)}{h} = f(x).$$

so that $A(x)$ is an anti-derivative of $f(x)$. General f is similar.

(b) Follows from (a).

15.4.3   Notation

We will write
$$F(x)|_a^b = [F(x)]_a^b = F(b) - F(a).$$

15.4.4   Properties of the definite integral

For some $c \in \mathbb{R}$, $f(x)$ continuous on $[a, b]$ there holds:

- $\int_b^a f(x)dx = - \int_a^b f(x)dx$

- $\int_a^b cdx = c(b - a)$

- $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

- $\int_a^b cf(x)dx = c \int_a^b f(x)dx$

- $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$

- $\int_a^a f(x)dx = 0$

15.4.5   Examples

(1) Show that $\ln x = \int_1^x \frac{1}{t}dt$, for $x > 0$.

$$
\begin{aligned}
\text{Set } g(x) &= \ln x - \int_1^x \frac{dt}{t} \text{ for } x > 0.\\
\text{WTS: } g(x) &= 0 \text{ for all } x > 0.\\
\text{So } g'(x) &= \frac{d}{dx}\ln x - \frac{d}{dx}\left[\int_1^x \frac{dt}{t}\right]\\
&= \frac{1}{x} - \frac{1}{x} \text{ by Fundamental Theorem of Calculus}\\
&= 0\\
\Rightarrow g(x) &= C, \text{ for some constant } C.\\
\Rightarrow g(x) &= g(1) = \ln 1 - \int_1^1 \frac{dt}{t} = 0
\end{aligned}
$$

(2) Evaluate $\int_1^2 \frac{1}{t} dt$.

$$
\begin{aligned}
\int_1^2 \frac{1}{t} dt &= [\ln t]_1^2 \\
&= \ln 2 - \ln 1 \\
&= \ln 2.
\end{aligned}
$$

(3) Evaluate $\int_0^{\pi/2} \sin x dx$.

$$
\begin{aligned}
\int_0^{\pi/2} \sin x dx &= [-\cos x]_0^{\pi/2} \\
&= -\cos \frac{\pi}{2} - (-\cos 0) \\
&= 0 + 1 \\
&= 1.
\end{aligned}
$$

(4) If $G(x) = \int_2^{\sin x} \frac{dt}{\sqrt{1+t^2}}$, find $G'(x)$.

$$
\begin{aligned}
\text{Let } F(u) &= \int_2^u \frac{dt}{\sqrt{1+t^2}} \\
\Rightarrow F'(u) &= \frac{1}{\sqrt{1+u^2}} \quad \text{(by the Fundamental Theorem of Calculus)}
\end{aligned}
$$

Now, $G(x) = F(\sin x) = F(u)$, with $u = \sin x$. By the chain rule,

$$
\begin{aligned}
G'(x) &= \frac{d}{dx} F(u) \\
&= \frac{dF(u)}{du} \cdot \frac{du}{dx} \\
&= \frac{1}{\sqrt{1+u^2}} \cos x \\
&= \frac{\cos x}{\sqrt{1+\sin^2 x}}.
\end{aligned}
$$

15.4.6   Example

Find the area between the x-axis and the curve $y = \sin x$ on $[0, \pi]$.

There holds $\sin x \geq 0$ on $[0, \pi]$, so

$$
\begin{aligned}
\text{Area} \; = \; \int_0^\pi \sin x \, dx \; &= \; [-\cos x]_0^\pi \\
&= \; -\cos \pi - (-\cos 0) \\
&= \; -(-1) + 1 \\
&= \; 2.
\end{aligned}
$$

15.4.7   Example

Find the area bounded by the curves $y = x^2$ and $y = x^3$ on the interval $[0, 1]$.

There holds $x^2 \geq x^3$ on $[0, 1]$,

$$
\begin{aligned}
\Rightarrow \; \text{Area} \; &= \; \int_0^1 x^2 dx - \int_0^1 x^3 dx \\
&= \; \left[\frac{1}{3} x^3\right]_0^1 - \left[\frac{1}{4} x^4\right]_0^1 \\
&= \; \frac{1}{3} - \frac{1}{4} \\
&= \; \frac{1}{12}.
\end{aligned}
$$

15.4.8   Corollary

If f is non-negative and continuous on $[a, b]$ then

$$
\int_a^b f(x) dx \geq 0.
$$

If f is non-positive and continuous on $[a, b]$ then

$$
\int_a^b f(x) dx = -(\text{area above graph below interval } [a, b]) \leq 0.
$$

15.4.9 Example

Find the area enclosed by the curves $y = x$ and $y = x^3$.

First, we need to determine where the two curves intersect:

$$x^3 = x \quad \Rightarrow \quad x^3 - x = 0 \Rightarrow x(x+1)(x-1) = 0$$
$$\Rightarrow \quad x = -1, 0, 1.$$

There holds: $x^3 \geq x$ on $[-1, 0]$ and $x \geq x^3$ on $[0, 1]$.

$$
\begin{aligned}
\Rightarrow \text{Area} \; &= \; \int_{-1}^{0} (x^3 - x)dx + \int_{0}^{1} (x - x^3)dx \\
&= \; \left[\frac{x^4}{4} - \frac{x^2}{2} \right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_{0}^{1} \\
&= \; -\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4} \\
&= \; \frac{1}{4} + \frac{1}{4} \\
&= \; \frac{1}{2}.
\end{aligned}
$$

15.5   Approximate integration (trapezoidal rule)

We can approximate the area under a curve by taking the sum of a finite number of rectangles under the graph as outlined in the previous section. If we choose the c_i to be the left endpoints of each rectangle, we obtain the approximation

$$\ell_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x.$$

If we choose the c_i to be the right endpoints of each rectangle we obtain the approximation

$$r_n = \sum_{i=1}^{n} f(x_i)\Delta x.$$

The trapezoidal rule is the approximation of the area under a curve which takes the average value of ℓ_n and r_n.

The formula for this approximation is therefore given by

$$
\begin{aligned}
\text{area } A \quad &\approx \quad \frac{\ell_n + r_n}{2} \\
&= \quad \frac{\Delta x}{2} \left[\sum_{i=1}^{n} (f(x_{i-1}) + f(x_i)) \right] \\
&= \quad \frac{\Delta x}{2} \left[(f(x_0) + f(x_1)) + (f(x_1) + f(x_2)) + \ldots + (f(x_{n-1}) + f(x_n)) \right] \\
&= \quad \frac{\Delta x}{2} \left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n) \right]
\end{aligned}
$$

where $\Delta x = (b - a)/n$ and $x_i = a + i\Delta x$.

The trapezoidal rule will appear in a Matlab module.

15.6 Improper integrals - Stewart, 6ed. pp. 544-551; 5ed. pp. 566-573

If $\int_a^t f(x)dx$ exists for every $t \geq a$, then

$$
\int_a^\infty f(x)dx = \lim_{t \to \infty} \int_a^t f(x)dx
$$

provided the limit exists.

If $\int_t^b f(x)dx$ exists for every $t \leq b$, then

$$
\int_{-\infty}^b f(x)dx = \lim_{t \to -\infty} \int_t^b f(x)dx
$$

provided the limit exists.

The integrals $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are called *convergent* if the limit exists and divergent otherwise.

If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx \quad \text{for any real } a.$$

Note that this must be independent of a. Note also that the usual properties of integrals apply provided all integrals are convergent.

15.6.1   Example

Find $\int_0^\infty e^{-x}dx$.

For $a > 0$,
$$\int_0^a e^{-x}dx = [-e^{-x}]_0^a = 1 - e^{-a}.$$
As $a \to \infty$, $1 - e^{-a} \to 1$, so
$$\int_0^\infty e^{-x}dx = \lim_{a\to\infty}(1 - e^{-a}) = 1.$$

15.6.2   Note

It is still possible to define the integral for some classes of functions which are not necessarily continuous: Stewart, 6ed. p. 548; 5ed. p. 570. Note also that there are other types of improper integral: Stewart, 6ed. p. 547; 5ed. p. 569.

15.7 Techniques of integration - Stewart, 6ed. ch. 8; 5ed. ch. 8

15.7.1 Substitution

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)dx = \int f(u)\frac{du}{dx}dx = \int f(u)du.$$

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

15.7.2 Example: Stewart, 6ed. p. 334 ex. 1; 5ed. p. 361 ex. 1

Find $\int x^3 \cos(x^4 + 2)dx$.

Substitute $u = x^4 + 2$ so $\frac{du}{dx} = 4x^3$ and $x^3 = \frac{\frac{1}{4}du}{dx}$.

$$
\begin{aligned}
\Rightarrow \int x^3 \cos(x^4 + 2)dx &= \int \frac{1}{4}\frac{du}{dx} \cos u \, dx \\
&= \int \frac{\cos u}{4} \, du \\
&= \frac{1}{4} \int \cos u \, du \\
&= \frac{1}{4} \sin u + c \\
&= \frac{1}{4} \sin(x^4 + 2) + c.
\end{aligned}
$$

15.7.3   Example: Stewart, 6ed. p. 337 ex. 7; 5ed. p. 364 ex. 7

Evaluate $\displaystyle\int_1^2 \frac{dx}{(5x-3)^2}$.

Substitute $u = 5x - 3$. Then $\dfrac{du}{dx} = 5$ and $dx = \dfrac{1}{5}du$.

Note also that $u(2) = 7$ and $u(1) = 2$. Therefore:

$$
\begin{aligned}
\int_1^2 \frac{dx}{(5x-3)^2} &= \int_2^7 \frac{1}{u^2} \cdot \frac{1}{5}du \\
&= \frac{1}{5}\int_2^7 u^{-2}du \\
&= \frac{1}{5}\left[-u^{-1}\right]_2^7 \\
&= \frac{1}{5}\left(-\frac{1}{7} + \frac{1}{2}\right) = \frac{1}{14}.
\end{aligned}
$$

15.7.4   Example: Trig substitution

Evaluate $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}}$.

Substitute $x = a\sin\theta$. Then $dx = a\cos\theta d\theta$. Hence:

$$
\begin{aligned}
\int \frac{dx}{\sqrt{a^2 - x^2}} &= \int \frac{a\cos\theta d\theta}{\sqrt{a^2 - a^2\sin^2\theta}} \\
&= \int \frac{a\cos\theta d\theta}{a\sqrt{1 - \sin^2\theta}} \\
&= \int \frac{\cos\theta}{\cos\theta}d\theta \\
&= \int d\theta \\
&= \theta + c \\
&= \arcsin\left(\frac{x}{a}\right) + c.
\end{aligned}
$$

15.7.5 Example

Evaluate $\displaystyle\int \cos x \sin x\, dx$.

Substitute $u = \sin x$. Then $\dfrac{du}{dx} = \cos x$. Hence:

$$
\begin{aligned}
\int \cos x \sin x\, dx &= \int u\, du \\
&= \frac{u^2}{2} + c \\
&= \frac{\sin^2 x}{2} + c.
\end{aligned}
$$

15.7.6 Example

Evaluate $\displaystyle\int (3x+1)(3x^2+2x)^3\, dx$.

Substitute $u = 3x^2 + 2x$. Then $\dfrac{du}{dx} = 6x + 2$ and $3x + 1 = \dfrac{1}{2} \cdot \dfrac{du}{dx}$. Hence:

$$
\begin{aligned}
\int (3x+1)(3x^2+2x)^3\, dx &= \int \frac{1}{2}\frac{du}{dx}u^3\, dx \\
&= \int \frac{u^3}{2}\, du \\
&= \frac{u^4}{8} + c \\
&= \frac{1}{8}(3x^2+2x)^4 + c.
\end{aligned}
$$

15.7.7 Integration by parts

Given two differentiable functions $u(x)$ and $v(x)$ we have a product rule for differentiation:

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx} = u'v + uv'.$$

Therefore uv is an antiderivative of $uv' + u'v$. Hence

$$
\begin{aligned}
uv &= \int (u'v + uv')\,dx \\[2mm]
&= \int u'v\,dx + \int uv'\,dx \\[2mm]
\Rightarrow \int uv'\,dx &= uv - \int u'v\,dx.
\end{aligned}
$$

The aim is to simplify the function that is integrated, i.e. $\int u'v\,dx$ should be easier to find than $\int uv'\,dx$. Also note that the antiderivative of v' needs to be found!

15.7.8 Example

Evaluate $\displaystyle\int x^3 \ln x\,dx$.

Set $u = \ln x$ and $v' = x^3$. Then $u' = \dfrac{1}{x}$ and $v = \dfrac{x^4}{4}$. Hence:

$$
\begin{aligned}
\int x^3 \ln x\,dx &= \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^4 \cdot \frac{1}{x}\,dx \\[2mm]
&= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3\,dx \\[2mm]
&= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + c.
\end{aligned}
$$

Note that if $u = x^3$ and $v' = \ln x$ is chosen, then $v = \int \ln x\,dx$ needs to be found.

15.7.9 Example

Evaluate $\int xe^x dx$.

Set $u = x$ and $v' = e^x$. Then $u' = 1$ and $v = e^x$.

$$
\begin{aligned}
\Rightarrow \int xe^x dx &= xe^x - \int e^x dx \\
&= (x-1)e^x + c.
\end{aligned}
$$

Note that for the choice $u = e^x$ and $v' = x$, the integral becomes more complicated than before.

15.7.10 General rule

- For $\int (\text{polynomial}) \cdot e^{ax+b} dx$, choose u to be the polynomial and $v' = e^{ax+b}$.

- For $\int (\text{polynomial}) \cdot \ln(ax+b) dx$, choose $u = \ln(ax+b)$ and v' to be the polynomial.

- Note that sometimes integration by parts needs to be applied more than once.

15.7.11 Example

Evaluate $\int x \sin x dx$.

Set $u = x$ and $v' = \sin x$. Then $u' = 1$ and $v = -\cos x$.

$$
\begin{aligned}
\Rightarrow \int x \sin x dx &= -x \cos x - \int -\cos x dx \\
&= -x \cos x + \sin x + c.
\end{aligned}
$$

15.8   Integrals involving $\ln$ function

If $y = \ln x$ for $x > 0$, then

$$\frac{dy}{dx} = \frac{1}{x} \Rightarrow \int \frac{dx}{x} = \ln x + c, \text{ for } x > 0.$$

For $x < 0$, if $y = \ln(-x)$ we have

$$\frac{dy}{dx} = \frac{-1}{-x} = \frac{1}{x},$$

so we have

$$\int \frac{dx}{x} = \ln(-x) + c, \text{ for } x < 0.$$

We often combine these cases and write

$$\int \frac{dx}{x} = \ln|x| + c, \quad x \neq 0.$$

15.9   Integration of rational functions by partial fractions

A ratio of two polynomials,

$$f(x) = \frac{P(x)}{Q(x)}$$

is called a *rational function*. Integration of a rational function is made possible by expressing it as the sum of simpler fractions, called *partial fractions*.

(In the three examples which follow, we will only consider rational functions for which the degree of $P(x)$ is less than the degree of $Q(x)$.)

The general process for obtaining the sum of partial fractions is as follows:

- Factor the denominator $Q(x)$ as much as possible;

- Express $f(x)$ as a sum of fractions each of which take either the form $\frac{A}{(ax+b)^i}$ or $\frac{Ax+B}{(ax^2+bx+c)^j}$, where the denominators of these fractions come from the factorisation of $Q(x)$.

Consider the following examples.

15.9.1   Example

Evaluate $\displaystyle\int \frac{x+2}{x^2+x}dx$.

The general technique for solving intergrals of the form $\displaystyle\int \frac{cx+d}{(x-a)(x-b)}$

is to write $\displaystyle\frac{cx+d}{(x-a)(x-b)} = \frac{A}{x-1} + \frac{B}{x-b}$.

In this case: $\displaystyle\frac{x+2}{x^2+x} = \frac{x+2}{x(x+1)}$.

$$
\begin{aligned}
\text{We want: } \frac{x+2}{x(x+1)} &= \frac{A}{x} + \frac{B}{x+1} \\
&= \frac{A(x+1) + Bx}{x(x+1)}. \\
\text{equate numerators } \Rightarrow x+2 &= A(x+1) + Bx \\
&= (A+B)x + A \quad \circledast \\
\text{equate coefficients } \Rightarrow (A+B) &= 1, \text{ and } A = 2. \\
\Rightarrow \quad B &= -1. \\
\text{Hence } \Rightarrow \int \frac{x+2}{x(x+1)}dx &= \int \frac{2}{x}dx - \int \frac{dx}{x+1} \\
&= 2\ln|x| - \ln|x+1| + C.
\end{aligned}
$$

An alternate method at $\circledast$ would be to pick convenient values of x.

i.e. $\quad x = -1 \Rightarrow B = -1 \qquad x = 0 \Rightarrow A = 2$

15.9.2 Example

Evaluate $\displaystyle\int \frac{dx}{x^2 - a^2}$, $a \neq 0$.

We write

$$
\begin{aligned}
\frac{1}{x^2 - a^2} &= \frac{1}{(x+a)(x-a)} = \frac{A}{x+a} + \frac{B}{x-a} \\
&= \frac{A(x-a) + B(x+b)}{(x+a)(x-a)} \\
&= \frac{(A+B)x - a(A-B)}{(x+a)(x-a)}
\end{aligned}
$$

Equating numerators and equating coefficients:

$$
\text{i.e.} \quad \left\{ \begin{array}{l} A + B = 0 \\ A - B = -\frac{1}{a} \end{array} \right.
$$

Hence

$$
\begin{aligned}
\Rightarrow A &= -\frac{1}{2a} \text{ and } B = \frac{1}{2a} \\
\Rightarrow \int \frac{dx}{x^2 - a^2} &= \frac{1}{2a} \int \left(-\frac{1}{x+a} + \frac{1}{x-a} \right) dx \\
&= \frac{1}{2a}\left(-\ln|x+a| + \ln|x-a| \right) + C.
\end{aligned}
$$

15.9.3 Example

Evaluate $\int \dfrac{2x^2 - x + 4}{x^3 + 4x} \, dx$.

Given that $x^3 + 4x = x(x^2 + 4)$, then

$$\frac{2x^2 - x + 4}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}.$$

Multiplying each side by $x^2 + 4$ we get:

$$2x^2 - x + 4 = (A + B)x^2 + Cx + 4A,$$

the solution to which is $A = 1$, $B = 1$ and $C = -1$. Hence:

$$\int \frac{2x^2 - x + 4}{x^3 + 4x} \, dx = \int \left(\frac{1}{x} + \frac{x - 1}{x^2 + 4} \right) dx$$

$$= \int \frac{1}{x} \, dx + \int \frac{x}{x^2 + 4} \, dx - \int \frac{1}{x^2 + 4} \, dx$$

1. The first integral is easy:

$$\int \frac{1}{x} \, dx = \ln |x| + c_1$$

2. For the second integral, set $u = x^2 + 4$.

Then $\dfrac{du}{dx} = 2x$ and $dx = \dfrac{du}{2x}$.

Hence:

$$\int \frac{x}{x^2 + 4} \, dx = \int \frac{x}{u} \cdot \frac{du}{2x}$$

$$= \frac{1}{2} \int \frac{1}{u} \, du$$

$$= \frac{1}{2} \ln |u| + c_2 = \frac{1}{2} \ln \left| x^2 + 4 \right| + c_2$$

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3. For the third integral, set $x = 2\tan\theta$. Then $dx = 2\sec^2\theta\, d\theta$. Hence

$$
\begin{aligned}
\int \frac{1}{x^2 + 4}\, dx &= \int \frac{2\sec^2\theta}{4\tan^2\theta + 4}\, d\theta \\
&= \int \frac{2\sec^2\theta}{4\sec^2\theta}\, d\theta \\
&= \frac{1}{2}\int d\theta \\
&= \frac{1}{2}\theta + c_3 \\
&= \frac{1}{2}\arctan\left(\frac{x}{2}\right) + c_3
\end{aligned}
$$

Thus

$$
\int \frac{2x^2 - x + 4}{x^3 + 4x}\, dx = \ln|x| + \frac{1}{2}\ln\left|x^2 + 4\right| - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + c.
$$

15.9.4   Rationalising substitutions

Some nonrational functions can be made rational through the use of an appropriate substitution. When the integrand contains an expression of the form $\sqrt[n]{g(x)}$, try using the substitution $u = \sqrt[n]{g(x)}$.

15.9.5   Example

Evaluate $\displaystyle\int \frac{\sqrt{x+4}}{x}\, dx$.

At this stage, the function is not rational, because the expression in the numerator is not a polynomial. As per the suggestion above, we make the substitution
$$u = \sqrt{x+4} = (x+4)^{1/2}.$$

$$\Rightarrow \quad x = u^2 - 4$$

$$\Rightarrow \quad \frac{du}{dx} = \frac{1}{2}(x+4)^{-1/2}$$

$$\Rightarrow \quad dx = 2(x+4)^{1/2}\, du = 2u\, du.$$

Hence
$$\int \frac{\sqrt{x+4}}{x}\, dx = 2\int \frac{u^2}{u^2 - 4}\, du$$

We now have a rational function. However, the degree of the polynomial in the numerator is no longer less than the degree of the polynomial in the denominator. In such cases, we take the preliminary step of dividing $Q(x)$ into $P(x)$ until a remainder $R(x)$ is obtained such that the degree of $R(x)$ is less than the degree of $P(x)$.

In this case, we get:
$$\frac{u^2}{u^2 - 4} = 1 + \frac{4}{u^2 - 4}.$$

$$
\begin{aligned}
2\int \frac{u^2}{u^2 - 4}\, du &= 2\int \left(1 + \frac{4}{u^2 - 4}\right) du \\[2mm]
&= 2u + 2\int \frac{4}{(u+2)(u-2)}\, du \\[2mm]
&= 2\sqrt{x+4} + 2\int \left(\frac{-1}{u+2} + \frac{1}{u-2}\right) du \\[2mm]
&= 2\sqrt{x+4} - 2\ln|u+2| + 2\ln|u-2| + c \\[2mm]
&= 2\sqrt{x+4} - 2\ln\left|\sqrt{x+4} + 2\right| + 2\ln\left|\sqrt{x+4} - 2\right| + c \\[2mm]
&= 2\sqrt{x+4} + 2\ln\left|\frac{\sqrt{x+4} - 2}{\sqrt{x+4} + 2}\right| + c.
\end{aligned}
$$

15.10   Volumes of revolution

Suppose $f(x)$ is positive and continuous on $[a, b]$. By rotating the graph of $f(x)$ above $[a, b]$ about the x-axis, we obtain a cylindrical solid. What is the volume of such a solid? This is what we call a *volume of revolution.*

(a)   (b)
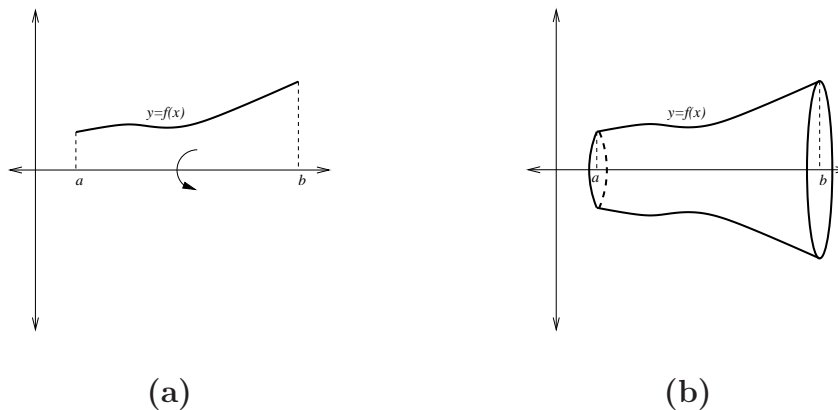
Figure 57: Rotating the curve in (a) about the x-axis gives the solid in (b).

15.10.1   Formula for volume

Suppose $f(x) \geq 0$ and continuous on $[a, b]$. The volume of revolution of the solid obtained by rotating the graph $y = f(x)$ above $[a, b]$ about the x-axis is

$$
V = \pi \int_a^b [f(x)]^2 dx.
$$

15.10.2   Outline of proof

For $x \in [a, b]$, let $V(x)$ be the volume obtained by rotating the graph of f above the interval $[a, x]$ about the x-axis. So $V(a) = 0$, $V(b) = V$. Then for $h \geq 0$ small

$$
\begin{aligned}
V(x + h) - V(x) \quad &= \quad \text{volume obtained by rotating graph} \\
&\qquad \text{of } f \text{ above } [x, x + h] \text{ about } x\text{-axis} \\
&\simeq \quad \text{volume of cylinder with radius } f(x) \text{ and height } h \\
&= \quad \pi [f(x)]^2 h \text{ (becomes exact as } h \to 0) \\
&\Rightarrow \quad \frac{V(x + h) - V(x)}{h} \simeq \pi [f(x)]^2
\end{aligned}
$$

Taking the limit $h \to 0$ we obtain

$$V'(x) = \pi[f(x)]^2$$

so the Fundamental Theorem yields

$$\int_a^b \pi[f(x)]^2 dx = [V(x)]_a^b = V(b) - V(a) = V(b) = V.$$

15.10.3   Example

Let $a > 0$. Find the volume of the solid obtained by rotating $y = f(x) = \sqrt{x}$ about the x-axis over $[0, a]$.

$$
\begin{aligned}
\text{volume} \quad &= \quad \pi \int_0^a [\sqrt{x}]^2 dx \\
&= \quad \pi \int_0^a x dx = \pi \left[\frac{x^2}{2}\right]_0^a \\
&= \quad \frac{1}{2}\pi a^2.
\end{aligned}
$$

15.10.4   Example

Determine the volume of the solid obtained by rotating $y = x$ about the x-axis over $[0, 1]$.

$$
\begin{aligned}
\text{volume} \quad &= \quad \pi \int_0^1 (x)^2 \, dx \\
&= \quad \frac{\pi}{3} \left[x^3\right]_0^1 \\
&= \quad \frac{\pi}{3}.
\end{aligned}
$$

15.10.5 Example

Determine the volume of the solid obtained by rotating $y = x^2$ about the x-axis over $[0, 1]$.

$$
\begin{aligned}
\text{volume} \quad &= \quad \pi \int_0^1 (x^2)^2 \, dx \\
&= \quad \pi \int_0^1 x^4 \, dx \\
&= \quad \frac{\pi}{5} \left[x^5 \right]_0^1 \\
&= \quad \frac{\pi}{5}.
\end{aligned}
$$

15.10.6 Example

The region enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resultant solid.

The curves $y = x$ and $y = x^2$ intersect at the points $(0, 0)$ and $(1, 1)$. Furthermore, $x \geq x^2$ on $[0, 1]$.

Hence, we can simply subtract the volume obtained by rotating $y = x^2$ about the x-axis over $[0, 1]$ from the volume obtained by rotating $y = x$ about the x-axis over $[0, 1]$:

$$
\begin{aligned}
\text{volume} \quad &= \quad \frac{\pi}{3} - \frac{\pi}{5} \\
&= \quad \frac{2\pi}{15}.
\end{aligned}
$$

Notes

16 Sequences

A sequence $\{a_n\}$ is a list of numbers with a definite order:

$$a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$$

In fact, sequences can be finite or infinite. Further, a sequence does not have to consist of *numbers*, but might contain any sort of mathematical structure, or indeed any other objects.

16.1 Formal Definition: Sequence

More formally, a sequence is a function, with domain being $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. We can also take the domain as $\mathbb{N} = \{1, 2, 3, \ldots\}$ and start the sequence at a_1 rather than a_0.

16.1.1 Motivation

Infinite sequences of numbers are useful in many applications. The sequence might represent approximations to the solution of a problem sduch as the bisection method. Or the sequence might represent a *time series*, where the numbers represent the population each year. Mathematically, we are interested in how the sequence behaves as the number of terms becomes large: does it converge to a solution, or how fast does the population grow? Sequences can also appear from the partial sums in an *infinite series*, which comprises our next major topic.

16.2 Representations

There are two main ways to represent the nth term in a sequence. Firstly, there is a *direct* (or *closed form* or *functional*) representation. Secondly, there is a *recursive* (or *indirect*) representation.

In the direct form, we can write

$$a_n = f(n).$$

In the recursive form, we write

$$a_n = f(a_{n-1}, a_{n-2}, \ldots, a_2, a_1, a_0).$$

The recursive form can have a simpler structure. It might be that $a_n = f(a_{n-1}, a_{n-2})$ or even $a_n = f(a_{n-1})$. Depending on the problem, it can be natural to express the formula in one or the other way.

16.2.1 Example: Newton's method

Newton's method is a numerical procedure to approximate the roots of functions. Most programs which find roots of functions are based on this method. For example, consider the equation

$$48x(1+x)^{60} - (1+x)^{60} + 1 = 0.$$

How would you find the roots of this equation? It is impossible to solve analytically (ie, precisely), but Newton's method gives us a way of approximating the roots to some degree of accuracy.

The idea behind the geometry of this method is given in Figure 58. We must have a starting guess (the value x_1), then we find the tangent line to the curve at that point, and find where the tangent line cuts the x-axis. This gives us the next point. We repeat the process until we have a close enough estimate.
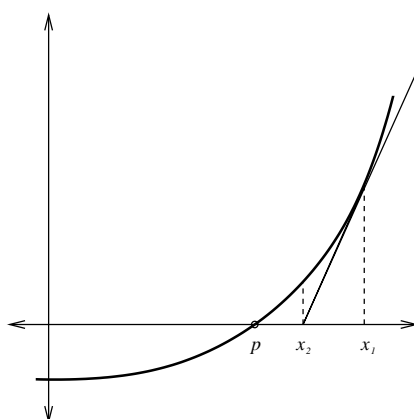
Figure 58: Starting at x_1, this gives you an idea of how Newton's method iterates to estimate the root p.

Mathematically, the slope of the tangent line to the curve at the point $y = f(x_1)$ is given by

$$f'(x_1) = \frac{f(x_1) - 0}{x_1 - x_2} \quad \Rightarrow \quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Repeating this procedure leads to the iterative formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

If the numbers x_n become closer and closer to p as n becomes large, we say the sequence *converges* to p.

This method is very sensitive to the type of curve and the choice of starting point.

16.2.2 Example

List the first terms of the sequence $a_n = \dfrac{n}{n+1}$.

$$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$$

16.2.3 Example

List the first terms of the recursive sequence $a_{n+1} = \dfrac{1}{3 - a_n}$ with $a_1 = 2$.

$$2, 1, \frac{1}{2}, \frac{2}{5}, \ldots$$

Note that for the recursive form, you must express one or more *initial conditions* or starting values. The recursive form is actually a *difference equation*, and there are similarities with differential equations that are studied in MATH1052.

16.2.4 Example: Fibonacci sequence

The Fibonacci sequence is defined through

$$a_n \quad = \quad f(a_{n-1}, a_{n-2}) = a_{n-1} + a_{n-2}$$

with $a_0 = 0$, $a_1 = 1$. List the first 9 terms.

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

16.3 Application: Rabbit population - Stewart, 6ed. p. 722 ex. 71; 5ed. p. 748 ex. 65

Suppose that rabbits live forever and that every month each pair produces a new pair which becomes productive at age 2 months. Starting with one newborn pair, how many pairs of rabbits will be there in the nth month?

Let a_n be the number of rabbit pairs in the nth month.

$$\Rightarrow \quad a_n = a_{n-1} + a_{n-2}, \quad \text{with } a_0 = 0, a_1 = 1.$$

16.4 Limits

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence. Then

$$\lim_{n \to \infty} a_n = \ell \quad , \quad \ell \in \mathbb{R}, \quad \text{means :}$$

a_n approaches ℓ as n gets larger and larger, ie. a_n is close to ℓ for n sufficiently large.

This is similar to $\lim_{x \to \infty} f(x) = \ell$, except here a_n is defined only for natural numbers n.

16.4.1 Convention

If a sequence $\{a_n\}_{n=0}^{\infty}$ has limit $\ell \in \mathbb{R}$, we say that a_n *converges* to ℓ and that the sequence $\{a_n\}_{n=0}^{\infty}$ is *convergent*. Otherwise the sequence is *divergent*.

16.4.2 Examples

Determine if the sequences $\{a_n\}$, with a_n as given below, are convergent and if so find their limits.

1. $a_n = \dfrac{1}{n}$

$$
\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{1}{n} \\
&= 0
\end{aligned}
$$

So $\{a_n\}$ converges to 0, i.e. $a_n \to 0$ as $n \to \infty$.

2. $a_n = \dfrac{1}{2} a_{n-1}$, $a_0 = -1$.

In direct form:
$$
a_n = -\frac{1}{2^n}, \quad \text{for } n = 0, 1, \ldots
$$

$$
\begin{aligned}
\Rightarrow \quad \lim_{n\to\infty} a_n &= \lim_{n\to\infty} \left(-\frac{1}{2^n} \right) \\
&= 0
\end{aligned}
$$

$\Rightarrow a_n \to 0$ as $n \to \infty$.

3. $a_n = (-1)^n$

The first few terms are: $1, -1, 1, -1, \ldots$. This sequence oscillates between ± 1 and therefore does not converge $\Rightarrow$ divergent sequence.

4. $a_n = r^n$, for $r = \dfrac{1}{2}, r = 1, r = 2$

$a_n = r^n$: 4 cases.

(1) $|r| < 1$: $\lim\limits_{n\to\infty} a_n = 0 \Rightarrow \{a_n\}$ converges to 0.

(2) $|r| > 1$: $\lim\limits_{n\to\infty} a_n = \infty \Rightarrow \{a_n\}$ diverges.

(3) $r = 1$: $\lim\limits_{n\to\infty} a_n = 1 \Rightarrow \{a_n\}$ converges to 1.

(4) $r = -1$: $\lim\limits_{n\to\infty} a_n$ does not exist $\Rightarrow \{a_n\}$ diverges.

So:

$r = \frac{1}{2}$: converges to 0.

$r = 1$: converges to 1.

$r = 2$: converges to 2.

16.5 Theorem: Limit laws

The following limit laws apply provided that the separate limits exist (that is $\{a_n\}$ and $\{b_n\}$ are convergent):

Suppose that $\{a_n\}$ and $\{b_n\}$ are convergent sequences such that

$$\lim_{n\to\infty} a_n = \ell \quad \text{and} \quad \lim_{n\to\infty} b_n = m$$

and c is a constant. Then

- $\lim\limits_{n\to\infty} (a_n \pm b_n) = \ell \pm m$;

- $\lim\limits_{n\to\infty} ca_n = c\ell$;

- $\lim\limits_{n\to\infty} (a_n b_n) = \ell m$; and

- $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\ell}{m}$, provided $m \neq 0$.

16.6 Theorem: Squeeze

If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ for some $n_0 \in \mathbb{N}$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = \ell$, then

---

$$\lim_{n\to\infty} b_n = \ell.$$

16.6.1 Example

Use the squeeze theorem on $\{a_n\}$, where $a_n = \dfrac{1}{n}\sin(n)$.

Since
$$-1 \leq \sin(n) \leq 1, \text{ for } n \geq 1$$
we have
$$-\frac{1}{n} \leq a_n \leq \frac{1}{n}, \text{ for } n \geq 1.$$
Now
$$\lim_{n\to\infty} -\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0,$$
so the squeeze theorem gives $\lim\limits_{n\to\infty} a_n = 0$.

16.7 Theorem: Continuous extension

Let f be a function on the real numbers such that $\lim\limits_{x\to\infty} f(x)$ exists. Let $f(n) = a_n$ for natural numbers n. Then:

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x).$$

16.7.1 Example

Evaluate $\lim\limits_{n\to\infty} \frac{\log n}{n}$.

Define $f(x) = \dfrac{\ln x}{x}$. Hence

$$
\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{\ln x}{x} \\
&= \lim_{x\to\infty} \frac{1/x}{1} \text{ by L'Hôpital's rule} \\
&= 0 \\
\Rightarrow \lim_{n\to\infty} \frac{\ln n}{n} &= 0.
\end{aligned}
$$

16.8   Useful sequences to remember

(1) For constant c, $\displaystyle\lim_{n\to\infty} c^n = \begin{cases} 0, & \text{if } |c| < 1 \\ 1, & \text{if } c = 1. \end{cases}$

Sequence $\{c^n\}_{n=0}^{\infty}$ is divergent if $c = -1$ or $|c| > 1$.

(2) For constant $c > 0$, $\displaystyle\lim_{n\to\infty} c^{1/n} = \lim_{n\to\infty} \sqrt[n]{c} = 1$.

(3) Even $\displaystyle\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} \sqrt[n]{n} = 1$.

(4) $\displaystyle\lim_{n\to\infty} \frac{1}{n!} = 0$.

(5) For constant c, $\displaystyle\lim_{n\to\infty} \frac{c^n}{n!} = 0$.

(6) $\displaystyle\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$.

(7) $\displaystyle\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = e^a$.

16.9   Application: Logistic sequence

Sequences which are bounded, but not convergent can be very interesting. The study of *chaos* appears in these sequences. A famous recursive sequence is the *logistic sequence* from population dynamics:

$$a_n = ka_{n-1}(1 - a_{n-1}), \text{ with } 0 \le a_{n-1} \le 1.$$

For certain values of the coefficient k, the behaviour is regular: convergent, or periodic. But other values of k yield chaotic sequences which cannot be predicted. You will explore the logistic equation in the Matlab Tutorials. Figure 59 has some examples of the logistic sequence when $k = 2.9, 3.5$ and 3.9.
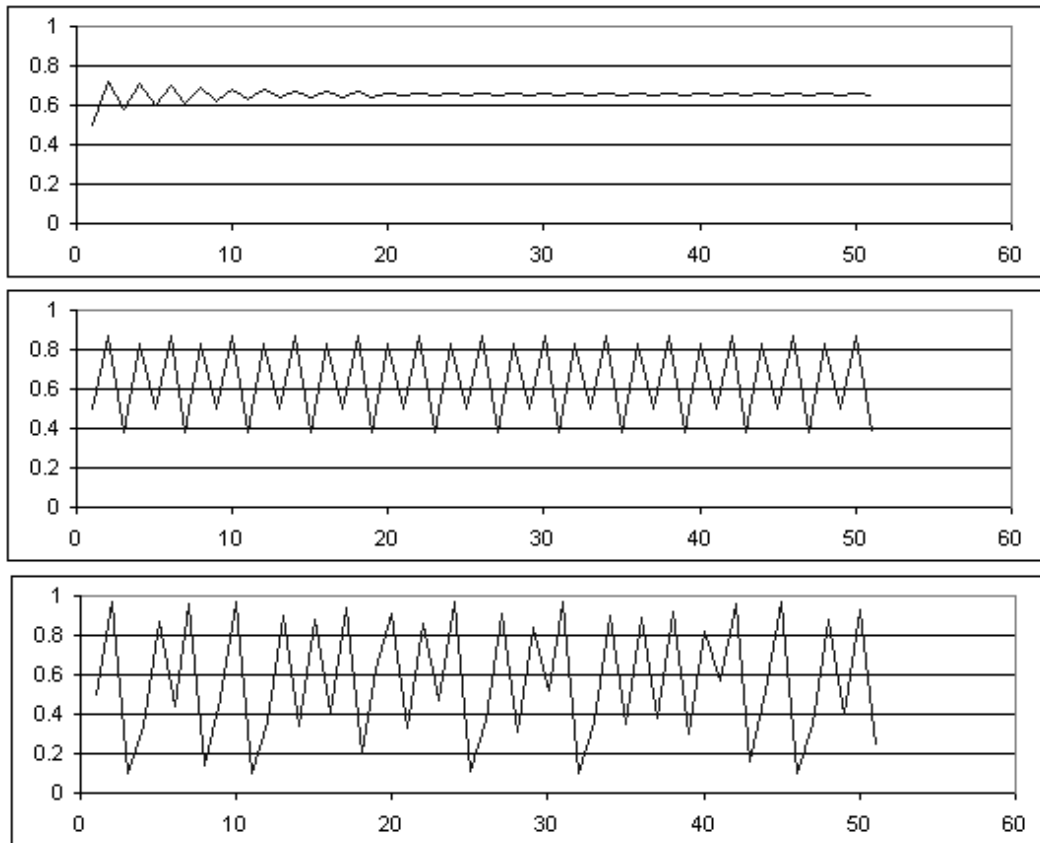
Figure 59: Logistic sequence, $a_0 = \frac{1}{2}$, $k = 2.9, 3.5, 3.9$

Notes

Notes

17 Series

17.1 Infinite sums (notation)

Recall that if we have an infinite sum we write

$$a_0 + a_1 + a_2 + \ldots + a_n + \ldots = \sum_{n=0}^{\infty} a_n.$$

Note that the lower bound ($n = 0$) of the sum may vary.

17.2 Motivation

Series come from many fields.

1. Approximation to problem solutions:

   - a_0 zeroth order approximation
   - $a_0 + a_1$ (a_1 small) first order approximation
   - $a_0 + a_1 + a_2$ (a_2 very small) second order approximation
   - $a_0 + a_1 + a_2 + \ldots + a_n$ nth order
   - $a_0 + a_1 + a_2 + \ldots + a_n + \ldots$ exact solution, provided the series converges

2. Current state of a process over infinite time horizon

3. Approximating function, e.g.,

$$\begin{aligned} e^x &= 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \ldots, \\ f(t) &= c_0 + c_1 \sin t + c_2 \sin(2t) + c_3 \sin(3t) + \ldots \end{aligned}$$

4. Measure the volume in a fractal process, like the amount of material in a botanic structure (Figure 60).
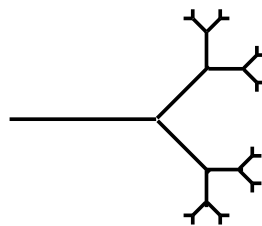
Figure 60: This is a graphical representation of a botanic fractal.

5. Riemann sums

17.3 Definition: Convergence - Stewart, 6ed. p. 724; 5ed. p. 750

Given a series $\sum\limits_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \ldots$, let s_n denote its nth partial sum:

$$
\begin{aligned}
s_0 &= a_0 \\
\\
s_1 &= a_0 + a_1 \\
\\
s_2 &= a_0 + a_1 + a_2 \\
&\vdots \\
\\
s_n &= a_0 + a_1 + \ldots + a_n = \sum_{k=0}^{n} a_k \, .
\end{aligned}
$$

If the sequence $\{s_n\}$ is convergent (i.e. $\lim\limits_{n\to\infty} s_n = s$ with $s \in \mathbb{R}$), then the series $\sum\limits_{n=0}^{\infty} a_n$ is said to be *convergent* and we write

$$
\lim_{n\to\infty} s_n = \sum_{n=0}^{\infty} a_n = s.
$$

The number s is called the *sum* of the series. Otherwise the series is said to be *divergent*.

17.3.1 Example

Does the series $\sum\limits_{n=0}^{\infty}(-1)^n$ converge or diverge?

$$
\begin{aligned}
s_0 &= (-1)^0 = 1 \\
s_1 &= (-1)^0 + (-1)^1 = 0 \\
s_2 &= 1 \quad \text{etc.}
\end{aligned}
$$

Therefore
$$
s_{2m} = 1 \quad \text{and} \quad s_{2m+1} = 0
$$
$$
\Rightarrow \lim_{n\to\infty} s_n \text{ does not exist} \Rightarrow \{s_n\} \text{ diverges}
$$

So $\sum\limits_{n=0}^{\infty}(-1)^n$ diverges.

17.3.2 Example

Show that the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}$ is convergent and find its sum.

First note that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$

Then

$$
\begin{aligned}
s_n &= \sum_{k=1}^{n} \frac{1}{k(k+1)} \\
&= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) \\
&= \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1} \right) \\
&= 1 - \frac{1}{n+1}.
\end{aligned}
$$

Therefore

$$
\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1} \right) = 1.
$$

Therefore the series is convergent and

$$
\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = 1.
$$

17.4   Theorem: p test

For $p \in \mathbb{R}$, the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is

convergent if $p > 1$

and divergent if $p \leq 1$.

Note the above sum is from $n = 1$. This is just a matter of taste since

$$
\sum_{n=1}^{\infty} \frac{1}{n^p} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^p}
$$

17.5 Theorem: nth term test - Stewart, 6ed. p. 728; 5ed. p. 754

If $\displaystyle\sum_{n=0}^{\infty} a_n$ is convergent then $\displaystyle\lim_{n\to\infty} a_n = 0$.

17.6 Test for divergence (nth term test)

If $\displaystyle\lim_{n\to\infty} a_n \neq 0$ then the series is divergent.

17.6.1 Caution

If $\displaystyle\lim_{n\to\infty} a_n = 0$ then we cannot say whether it is convergent or divergent.

For example,

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

but the *harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

diverges (*p*-series with $p = 1$).

17.6.2 Examples

Verify that these series diverge by the nth term test.

(i) $\displaystyle\sum_{n=0}^{\infty} (-1)^n$

With $a_n = (-1)^n$, $\displaystyle\lim_{n\to\infty} a_n \neq 0 \Rightarrow$ series is divergent.

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{5n^2 + 4}$

$$
\begin{aligned}
\lim_{n\to\infty} \frac{n^2}{5n^2 + 4} &= \lim_{n\to\infty} \frac{1}{5 + \frac{4}{n^2}} \\
&= \frac{1}{5} \neq 0
\end{aligned}
$$

$\Rightarrow$ series is divergent.

17.7   Example: Geometric series

The series

$$
\sum_{n=0}^{\infty} ar^n
$$

is convergent if $|r| < 1$ with $\displaystyle\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ and divergent if $|r| \geq 1$.

17.7.1   Proof

Firstly, let's prove that

$$
s_n = \begin{cases} a + ar + \cdots + ar^n = \frac{a\left(1 - r^{n+1}\right)}{1-r}, & \text{if } r \neq 1 \\ (n+1)a, & \text{if } r = 1 \end{cases}
$$

Let $s_n = \displaystyle\sum_{k=0}^{n} ar^k$. Then

$$
\begin{aligned}
s_n &= a + ar + ar^2 + \ldots + ar^n \\
\text{and} \quad rs_n &= ar + ar^2 + \ldots + ar^n + ar^{n+1}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
rs_n - s_n = (r-1)s_n &= \sum_{k=1}^{n+1} ar^k - \sum_{k=0}^{n} ar^k \\
&= ar^{n+1} - a \\
&= a\left(r^{n+1} - 1\right).
\end{aligned}
$$

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Hence

$$
\begin{aligned}
s_n &= \frac{a\left(r^{n+1} - 1\right)}{r - 1} \\
&= \frac{a\left(1 - r^{n+1}\right)}{1 - r}, \quad \text{for } r \neq 1.
\end{aligned}
$$

We now consider four cases.

$\boxed{\text{case 1:} \quad |r| < 1}$

$$
\begin{aligned}
\lim_{n\to\infty} s_n &= \lim_{n\to\infty} \frac{a\left(1 - r^{n+1}\right)}{1 - r} \\
&= \frac{a}{1 - r} \quad \left(\text{since } \lim_{n\to\infty} r^{n+1} = 0, \text{ when } |r| < 1\right)
\end{aligned}
$$

$\boxed{\text{case 2:} \quad |r| > 1}$ $\quad \lim_{n\to\infty} s_n$ does not exist. Hence the series is divergent.

$\boxed{\text{case 3:} \quad r = 1}$ $\quad$ In this case, $s_n = (n+1)a$, and

$$
\lim_{n\to\infty} s_n = \lim_{n\to\infty} (n+1)a
$$

does not exist. Hence the series is divergent.

$\boxed{\text{case 4:} \quad r = -1}$ $\quad$ In this case, $s_n = a\sum_{k=0}^{n}(-1)^k$. Hence $\lim_{n\to\infty} s_n$ does not exist and therefore the series is divergent.

17.7.2 Intuition

If all the terms in the series stay "large", then the series is divergent. If the individual terms are becoming smaller then the series *may* converge, but not necessarily.

17.7.3 Application: Plant structure

The following is a model with an infinite amount of recursion. We start with one branch of length L and radius R. At the end of each branch of length ℓ and radius r, we add three branches of length $\ell/2$ and radius $r/3$. After a few iterations we obtain a diagram like Figure 61.
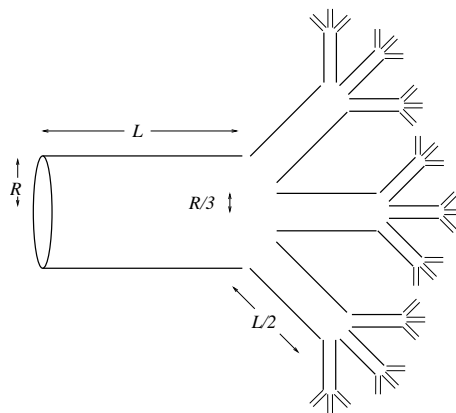
Figure 61: This is the first few iterations of our model of a plant structure. Note that this picture should really be visualised in 3 dimensions.

In fact, this type of model is used by experts to model trees, for example the franjipani, as in Figure 62.

Figure 62: This is a model of a franjipani similar to our simple model of plant structure.

What is the volume of the infinitely iterated structure? Remember the formula for the volume of a cylinder of radius r and length ℓ is $\pi r^2 \ell$.

The volume is given by the sum

$$
\begin{aligned}
V &= \pi r^2 \ell + 3\pi \left(\frac{r}{3}\right)^2 \left(\frac{\ell}{2}\right) + 9\pi \left(\frac{r}{9}\right)^2 \left(\frac{\ell}{4}\right) + 27\pi \left(\frac{r}{27}\right)^2 \left(\frac{\ell}{8}\right) + \dots \\
&= \pi r^2 \ell \left[1 + \frac{1}{6} + \frac{1}{6^2} + \frac{1}{6^3} + \dots\right]
\end{aligned}
$$

which is a geometric series with $a = \pi r^2 \ell$ and $r = 1/6$. Therefore it converges to

$$
\begin{aligned}
V &= \frac{a}{1-r} \\
&= \frac{\pi r^2 \ell}{1 - \frac{1}{6}} \\
&= \frac{6\pi r^2 \ell}{5}.
\end{aligned}
$$

17.8 Application: Bouncing ball - Stewart, 6ed. p. 731 qn 58; 5ed. p. 756 qn 52

A certain ball has the property that each time it falls from a height h onto a hard, level surface, it rebounds to a height rh, where $0 < r < 1$. Suppose that the ball is dropped from an initial height of H metres.

(a) Find the total distance that the ball travels.

Initially, the ball falls a distance H, then rebounds a distance rH, falls rH, rebounds r^2H, falls r^2H, etc. The total distance it travels is

$$
\begin{aligned}
H + 2rH + 2r^2H + 2r^3H + \dots &= H(1 + 2r + 2r^2 + 2r^3 + \cdots) \\
&= H(1 + 2r(1 + r + r^2 + \dots)) \\
&= H\left[1 + 2r\left(\frac{1}{1-r}\right)\right] \\
&= H\left(\frac{1+r}{1-r}\right) \text{ metres.}
\end{aligned}
$$

(b) Calculate the total time that the ball travels. Use the fact that the ball falls $gt^2/2$ metres in t seconds (from classical physics).

For each bounce, the time taken depends on the height from which the ball falls. For a height $h_i = \dfrac{gt_i^2}{2}$, $i = 1, 2, \ldots$, then $t_i = \sqrt{\dfrac{2h_i}{g}} = \sqrt{\dfrac{2r^i H}{g}}$. Thus the total time is given by

$$
\begin{aligned}
\text{total time} \;&=\; \sqrt{\frac{2H}{g}} + 2\sqrt{\frac{2rH}{g}} + 2\sqrt{\frac{2r^2 H}{g}} + 2\sqrt{\frac{2r^3 H}{g}} + \ldots \\[2mm]
&=\; \sqrt{\frac{2H}{g}}\left(1 + 2\sqrt{r} + 2\sqrt{r^2} + 2\sqrt{r^3} + \cdots\right) \\[2mm]
&=\; \sqrt{\frac{2H}{g}}\left(1 + 2\sqrt{r}\left(1 + \sqrt{r} + \sqrt{r^2} + \cdots\right)\right) \\[2mm]
&=\; \sqrt{\frac{2H}{g}}\left(1 + 2\sqrt{r}\left(\frac{1}{1 - \sqrt{r}}\right)\right) \\[2mm]
&=\; \sqrt{\frac{2H}{g}}\left(\frac{1 + \sqrt{r}}{1 - \sqrt{r}}\right).
\end{aligned}
$$

17.8.1   Example: Repeated drug dosage

A person with an ear infection is prescribed a treatment of ampicillin (a common antibiotic) to be taken four times a day in 250mg doses. It is known that at the end of a six hour period, about 4% of the drug is still in the body (since the drug is being excreted by the body between doses). What quantity of drug is in the body after the tenth tablet? The fortieth? Predict what quantity of drug will be in the body after long term treatment.

Let Q_n represent the quantity of ampicillin in the blood right after the nth tablet. Then

$$
\begin{aligned}
Q_1 &= 250 \\
Q_2 &= 250(1 + 0.04) \\
&= 260 \\
Q_3 &= 250\left(1 + 0.04 + 0.04^2\right) \\
&= 260.4 \\
Q_n &= 250(1 + 0.04 + (0.04)^2 + (0.04)^3 + \ldots + (0.04)^{n-1})
\end{aligned}
$$

Note that the pattern is a geometric series with $a = 250$ and $r = 0.04$.

Therefore

$$
\lim_{n \to \infty} Q_n = \frac{250}{1 - 0.04} \approx 260.42 \text{mg}.
$$

The amount of ampicillin in the blood appears to stabilize.

$$
\begin{aligned}
\text{Note} \quad Q_{10} &= 260.416666666664 \\
Q_{40} &= 260.416666666667 \\
\text{so} \quad Q_{40} - Q_{10} &= \sum_{n=11}^{40} 250(0.04)^{n-1} \\
&= 2.73 \times 10^{-12} \text{ mg}.
\end{aligned}
$$

17.9 Convergence theorem and limit laws

If $\sum\limits_{n=0}^{\infty} a_n = \ell$ and $\sum\limits_{n=0}^{\infty} b_n = m$ are convergent series, then so are the series $\sum\limits_{n=0}^{\infty} ca_n$ (where c is a constant), $\sum\limits_{n=0}^{\infty} (a_n \pm b_n)$ and

$$\sum_{n=0}^{\infty} ca_n = c \sum_{n=0}^{\infty} a_n$$

$$\sum_{n=0}^{\infty} (a_n \pm b_n) = \sum_{n=0}^{\infty} a_n \pm \sum_{n=0}^{\infty} b_n$$

17.9.1 Example

Evaluate $\sum\limits_{n=0}^{\infty} x^n (2x + 1)$, for $|x| < 1$.

$$\sum_{n=0}^{\infty} x^n (2x + 1) = (2x + 1) \sum_{n=0}^{\infty} x^n$$

Now $\sum\limits_{n=0}^{\infty} x^n$ is a convergent geometric series (since $|x| < 1$) with $a = 1$. Hence

$$(2x + 1) \sum_{n=0}^{\infty} x^n = (2x + 1) \frac{1}{1 - x}$$

$$= \frac{2x + 1}{1 - x}.$$

17.10   Comparison tests

If we are trying to determine whether or not $\sum_{n=0}^{\infty} a_n$ converges, where a_n contains complicated terms, then it may be possible to bound the series by using simpler terms. If $|a_n|$ is always smaller than $|b_n|$ and $\sum_{n=0}^{\infty} |b_n|$ converges, then $\sum_{n=0}^{\infty} a_n$ should also. On the other hand, if a_n is always large - in fact larger than $|b_n|$, and $\sum_{n=0}^{\infty} b_n$ diverges, then $\sum_{n=0}^{\infty} a_n$ should diverge too.

17.11   Theorem: Comparison test - Stewart, 6ed. p. 741; 5ed. p. 767

Suppose that $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are series with all non-negative terms.

---

1. If $\sum_{n=0}^{\infty} b_n$ is convergent and $0 \leq a_n \leq b_n$ for all n, then $\sum_{n=0}^{\infty} a_n$ is also convergent.

2. If $\sum_{n=0}^{\infty} b_n$ is divergent and $a_n \geq b_n \geq 0$ for all n, then $\sum_{n=0}^{\infty} a_n$ is also divergent.

Note that this applies for infinite series starting at any n ($= 1, 2$, etc).

17.11.1   Example

Test the series $\sum_{n=1}^{\infty} \dfrac{\ln n}{n}$ for convergence.

Note that for $n \geq 3$, $\ln(n) > 1$. Hence $\dfrac{\ln(n)}{n} > \dfrac{1}{n}$,   for $n \geq 3$. Since $\sum_{n=1}^{\infty} \dfrac{1}{n}$ is divergent (p-series with $p = 1$), by the comparison test $\sum_{n=1}^{\infty} \dfrac{\ln n}{n}$ is divergent too.

17.11.2 Example

Test the series $\displaystyle\sum_{n=0}^{\infty} \frac{5}{2 + 3^n}$ for convergence.

> Note that
> $$0 < \frac{5}{2 + 3^n} < \frac{5}{3^n}, \quad \text{for all } n \geq 0.$$
> Now
> $$\sum_{n=0}^{\infty} \frac{5}{3^n} = \sum_{n=0}^{\infty} 5\left(\frac{1}{3}\right)^n$$
> is a convergent geometric series with $a = 5$ and $r = \dfrac{1}{3}$. Hence, by the comparison test, $\displaystyle\sum_{n=0}^{\infty} \frac{5}{2 + 3^n}$ is also convergent.

17.12 Alternating series

The tests for convergence have so far required terms to be positive. However, if terms alternate in sign $+, -, +, -, \ldots$ (or $-, +, -, +, \ldots$) there are some special results. We call such a series an *alternating series*.

Here is quite an interesting result:

We have seen that the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

is divergent. However, the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

is convergent. Why?

17.13 Theorem: Alternating series test - Stewart, 6ed. p. 746; 5ed. p. 772

If the alternating series

$$\sum_{n=0}^{\infty}(-1)^n b_n = b_0 - b_1 + b_2 - b_3 + \ldots \text{ (all } b_n > 0)$$

satisfies

$$
\begin{array}{ll}
\text{(i)} & b_{n+1} \leq b_n, \text{ for all } n \geq 0 \text{ and} \\
\text{(ii)} & \lim_{n \to \infty} b_n = 0
\end{array}
$$

then the series is convergent.

17.13.1 Example

$$\sum_{n=0}^{\infty}(-1)^n \frac{1}{\sqrt{n+1}}$$

Clearly,

$$b_n = \frac{1}{\sqrt{n+1}} > \frac{1}{\sqrt{n+2}} = b_{n+1}, \text{ for all } n \geq 0,$$

and

$$
\begin{aligned}
\lim_{n \to \infty} b_n &= \lim_{n \to \infty} \frac{1}{\sqrt{n+1}} \\
&= 0.
\end{aligned}
$$

Therefore, by the alternating series test $\sum_{n=0}^{\infty}(-1)^n \frac{1}{\sqrt{n+1}}$ coverges.

17.13.2 Example

$$\sum_{n=1}^{\infty}(-1)^n \frac{3n}{4n-1}$$

In this case $b_n = \dfrac{3n}{4n-1}$.

$$
\begin{aligned}
\lim_{n\to\infty} b_n &= \lim_{n\to\infty}\left(\frac{3n}{4n-1}\right) \\
&= \lim_{n\to\infty}\left(\frac{3}{4-1/n}\right) \\
&= \frac{3}{4}
\end{aligned}
$$

Hence, the series diverges by the nth term test.

17.13.3 Example

$$1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \ldots$$

$$
\begin{aligned}
1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \ldots &= \left(1 + \frac{1}{2}\right) - \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6}\right) - \ldots \\
&= \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2n+1} + \frac{1}{2n+2}\right)
\end{aligned}
$$

Consequently, this is an alternating series with $b_n = \dfrac{1}{2n+1} + \dfrac{1}{2n+2}$. Clearly,

$$b_n > b_{n+1} \quad \left(= \frac{1}{2n+4} + \frac{1}{2n+3}\right),$$

for all $n \geq 0$.

Furthermore,

$$
\begin{aligned}
\lim_{n\to\infty} b_n &= \lim_{n\to\infty}\left(\frac{1}{2n+1} + \frac{1}{2n+2}\right) \\
&= 0
\end{aligned}
$$

Hence, by the alternating series test, this series converges.

17.14 Definition: Absolute/conditional convergence

A series $\sum\limits_{n=0}^{\infty} a_n$ is called *absolutely convergent* if the series $\sum\limits_{n=0}^{\infty} |a_n|$ is convergent.

A series $\sum\limits_{n=0}^{\infty} a_n$ is called *conditionally convergent* if it is convergent, but not absolutely convergent.

17.14.1 Examples

Are the following series conditionally or absolutely convergent?

$$(i) \quad \sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$$

$\sum\limits_{n=1}^{\infty}(-1)^n\dfrac{1}{n}$ is an alternating series with $b_n = \dfrac{1}{n}$. Since $b_n > b_{n+1}$, for all $n \geq 1$, and $\lim\limits_{n\to\infty} b_n = 0$, then by the alternating series test, this series converges.

Now let $a_n = (-1)^n\dfrac{1}{n}$, for $n \geq 1$. Then

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n},$$

and this series is divergent (*p*-series with $p = 1$).

Consequently, the series is conditionally convergent.

$$(ii) \quad \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^4}$$

Let $a_n = (-1)^n \dfrac{1}{n^4}$. Then

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^4},$$

and this series converges (p-series with $p = 4$).

Hence the series is absolutely convergent.

$$(iii) \quad \sum_{n=1}^{\infty} \frac{\cos n}{n^2}$$

Let $a_n = \dfrac{\cos n}{n^2}$ and note that

$$-1 \leq \cos n \leq 1 \quad \Rightarrow \quad -\frac{1}{n^2} \leq \frac{\cos n}{n^2} \leq \frac{1}{n^2}$$
$$\Rightarrow \quad \frac{|\cos n|}{n^2} \leq \frac{1}{n^2}.$$

Note also that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series with $p = 2$). Consequently, by the comparison test,

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$$

converges.

Consequently, the series is absolutely convergent.

17.14.2 Theorem: Absolute convergence

If a series is absolutely convergent, then it is convergent.

17.14.3 Proof

Observe that
$$0 \quad \leq \quad a_n + |a_n| \quad \leq \quad 2|a_n|$$
$\therefore$ If a series is absolutely convergent then $\sum_n |a_n|$ converges and hence so too

does $\quad \sum_n 2|a_n| = 2 \sum_n |a_n|$.

$\therefore$ By the comparison test
$$\sum_n (a_n + |a_n|)$$
also converges. Thus
$$\sum_n a_n = \sum_n (a_n + |a_n|) - \sum_n |a_n|$$
must converge.

17.15 Theorem: Ratio test - Stewart, 6ed. p. 752; 5ed. p. 778

A powerful test for convergence of series is the ratio test. We will use it extensively in the following sections.

1. If $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

2. If $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \infty$, then the series $\sum\limits_{n=1}^{\infty} a_n$ is divergent.

3. If $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = 1$, then the ratio test is inconclusive.

4. If $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right|$ is not defined, then the ratio test is inconclusive.

Note that the ratio test only tests for absolute convergence. The ratio test does not give any estimates of the sum.

17.15.1 Examples

Use the ratio test to determine, if possible, whether the following series are absolutely convergent.

$$(i) \quad \sum_{n=0}^{\infty} ar^n$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{ar^{n+1}}{ar^n} \right|$$

$$= \lim_{n\to\infty} |r|$$

$$= |r|$$

which is
$$\begin{cases} < 1, & |r| < 1 \\ = 1, & |r| = 1 \\ > 1, & |r| > 1 \end{cases}$$

Hence the series is:

- absolutely convergent if $|r| < 1$; and

- divergent if $|r| > 1$.

If, however, $r = 1$, then the ratio test gives us no information.

Note: this is a geometric series, see Section 17.7.

$$(ii) \quad \sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{(-1)^n n^3} \right|$$

$$= \lim_{n\to\infty} \left| \frac{-1(n+1)^3}{3n^3} \right|$$

$$= \lim_{n\to\infty} \frac{1}{3} \left(\frac{n+1}{n} \right)^3$$

$$= \frac{1}{3} \lim_{n\to\infty} \left(1 + \frac{1}{n} \right)^3$$

$$= \frac{1}{3} < 1$$

Hence, by the ratio test, this series is absolutely convergent.

$$(iii) \quad \sum_{n=1}^{\infty} \frac{1}{n!} x^n$$

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{x}{n+1} \right|$$

$$= |x| \lim_{n \to \infty} \frac{1}{n+1}$$

$$= |x| \cdot 0$$

$$= 0 < 1$$

$\therefore$ Series is absolutely convergent for all x.

$$(iv) \quad \sum_{n=0}^{\infty} (2x+5)^n$$

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(2x+5)^{n+1}}{(2x+5)^n} \right|$$

$$= |2x+5|$$

$\Rightarrow$ Series converges absolutely when $|2x+5| < 1$ (ie when $-3 < x < -2$) and diverges when $|2x+5| > 1$ (ie when $x < -3$ or $x > -2$).

Note: the ratio test gives no information about convergence of the series when $x = -3, \ -2$ (In fact, the series diverges at both these points).

Notes

18 Power series and Taylor series

18.1 Definition: Power series

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$$

This is a *power series about a*. a and c_n's are fixed, x variable.

Such series have many applications, for example, solutions of higher order O.D.E's (ordinary differential equations) with non-constant coefficients, which in turn are very important to many areas of science and engineering.

18.1.1 Example: Bessel function

Use the ratio test to determine for which values of x the *Bessel function of order zero* (denoted $J_0(x)$) converges, where

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

$x = 0$: converges.

$x \neq 0$:

$$
\begin{aligned}
\left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{(-1)^{n+1} x^{2n+2}}{2^{2n+2}[(n+1)!]^2} \cdot \frac{2^{2n}(n!)^2}{(-1)^n x^{2n}} \right| \\
&= \left| \frac{(-1)x^2}{2^2(n+1)^2} \right| \\
&= \frac{x^2}{4(n+1)^2} \\
&\to \quad 0 \text{ as } n \to \infty \text{ for all } x.
\end{aligned}
$$

So by the ratio test, the series converges absolutely for all x.

Bessel functions first arose when Bessel solved Kepler's equation for describing planetary motion. Since then, they have been used in applications such as temperature distribution on a circular plate and the shape of a vibrating membrane (such as the human ear).

The graph of $J_0(t)$ is given in Figure 63.
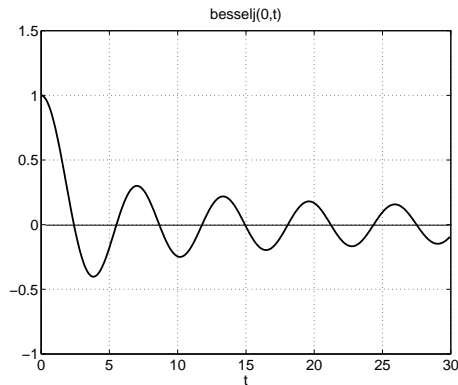
Figure 63: The Bessel function $J_0(t)$ is represented as `besselj(0,t)` in MAT-LAB.

18.2 Theorem: Defines radius of convergence - Stewart, 6ed. p. 752; 5ed. p. 787

For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

---

1. The series converges only when $x = a$.

2. The series converges for all x.

3. There is a positive number r (called the *radius of convergence*) such that the series converges if $|x - a| < r$ and diverges if $|x - a| > r$.

In case (1), we say that the radius of convergence is 0, while in case (2) we say that the radius of convergence is ∞.

The *interval of convergence* of a power series is the interval that consists of all values of x for which the series converges. In case (1) the interval is a single point, a. In case (2) the interval is $(-\infty, \infty)$. In case (3) the interval is either $(a - r, a + r)$, $(a - r, a + r]$, $[a - r, a + r)$ or $[a - r, a + r]$.

18.3 Radius of convergence

The series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence

$$r = \lim_{n \to \infty} \left| \frac{c_n}{c_{n+1}} \right|,$$

provided that the limit is defined. An easier approach is to set
$\Lambda = \lim_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right|$, and put $r = \frac{1}{\Lambda}$ (with $r = 0$ if $\Lambda = \infty$ and $r = 0$ if $\Lambda = 0$).

If $\left| \dfrac{c_n}{c_{n+1}} \right| \to \infty$, as $n \to \infty$, then the radius of convergence is $r = \infty$. That is, the series converges for all x.

18.3.1 Proof

Series is $\sum_{n=0}^{\infty} a_n$, $a_n = c_n(x-a)^n$.

$$
\begin{aligned}
\left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n} \right| \\
&= \left| \frac{c_{n+1}}{c_n} \right| |x-a| \\
\Rightarrow \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| &= |x-a| \lim_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right| \\
&= |x-a| \Lambda \\
\Rightarrow \mathrm{L} = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| &= |x-a| \Lambda
\end{aligned}
$$

By the ratio test, the series converges if L< 1, i.e. $|x-a| \Lambda < 1$,
i.e. $|x-a| \frac{1}{r} < 1$, i.e. $|x-a| < r$.

Similarly, the series diverges if $\mathrm{L} = |x-a| \Lambda > 1$, i.e. $|x-a| > r$.

This result does not apply when the sequence $\left| \dfrac{c_n}{c_{n+1}} \right|$ is *not defined*, but we may still get the radius of convergence by the ratio test.

18.3.2 Example

What is the radius of convergence for the series $\sum_{n=0}^{\infty} \dfrac{(-3)^n x^n}{\sqrt{n+1}}$? When does the series converge/diverge?

The series is $\sum_{n=0}^{\infty} c_n x^n$, $c_n = \dfrac{(-3)^n}{\sqrt{n+1}}$.

$$
\begin{aligned}
\Rightarrow \left| \frac{c_{n+1}}{c_n} \right| &= \left| \frac{(-3)^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n} \right| \\
&= 3 \frac{\sqrt{n+1}}{\sqrt{n+2}} \\
&\to 3 \text{ as } n \to \infty \\
\Rightarrow \Lambda &= \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = 3 \\
\Rightarrow r &= \frac{1}{\Lambda} = \frac{1}{3}
\end{aligned}
$$

So the series converges absolutely for $|x| < \frac{1}{3}$ and diverges for $|x| < \frac{1}{3}$.

Remark: in fact the series also diverges at $x = \pm\frac{1}{3}$.

18.4 Taylor series

Given a function $f(x)$ it is sometimes possible to expand $f(x)$ as a power series

$$
f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots
$$

How do we find the coefficients c_n in general? Let us assume that f and all its derivatives are defined at $x = a$.

First note that
$$
f(a) = c_0.
$$

Now take the derivative of both sides. Assuming that we are justified in differentiating term by term on the right-hand side, this gives

$$
f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots
$$

so that
$$f'(a) = c_1.$$

Taking the derivative again and again gives

$$\begin{aligned} f''(a) &= 2c_2 & \Rightarrow && c_2 &= f''(a)/2 \\ f'''(a) &= 6c_3 & \Rightarrow && c_3 &= f'''(a)/3! \end{aligned}$$

and, in general

$$c_n = f^{(n)}(a)/n!, \text{ with } f^{(0)}(x) \equiv f(x).$$

This leads to

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The right-hand side is called the *Taylor series* of $f(x)$ about $x = a$. So this result says that *if $f(x)$ has a power series representation (expansion) at $x = a$*, then it must be a Taylor series.

18.4.1   Example: MacLaurin series

When $a = 0$ in the Taylor series we call the series a *MacLaurin series*.

18.4.2   Note

The function f and all its derivatives must be defined at $x = a$ for the series to exist. In this case we say that f is infinitely differentiable at $x = a$.

18.5   Theorem: Formula for Taylor series

Suppose $f(x)$ is infinitely differentiable at $x = a$. Then for f "sufficiently well behaved",

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \text{ for } |x-a| < r,$$

where r is the radius of convergence.

(Note that if we define a sequence of numbers

$$b_n = \max_{x \in [-r,r]} |f^{(n)}(x)|, \ n \in \mathbb{N},$$

then the "sufficiently well behaved" means

$$\lim_{n \to \infty} \frac{b_n r^n}{n!} = 0.$$

Unless otherwise noted, all functions we consider in this section are "sufficiently well behaved".)

For $|x - a| < r$, we have approximately

$$f(x) \simeq \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x - a)^n$$

which is a useful polynomial approximation. The smaller $|x - a|$, the better the approximation. This approximation can be made accurate to arbitrary order by taking k sufficiently large provided $|x - a| < r$.

This begins to address the question of whether or not $f(x)$ has a power series representation. The details are beyond the scope of this course, but for those interested there is a good discussion about this issue and Taylor's inequality in Stewart, 6ed. pp. 772-774; 5ed. pp. 798-800.

18.5.1   Example

Find the Taylor series of $f(x) = \ln x$ about $x = 1$. Determine its radius of convergence.

$$
\begin{aligned}
f^{(0)}(x) &= f(x) = \ln x \\
f'(x) &= \frac{1}{x} = x^{-1}, \ f''(x) = -x^{-2}, \ f'''(x) = 2x^{-3}, \text{ and so on} \\
n \geq 2 : \ f^{(n)}(x) &= (-1)^{n-1}(n-1)! \ x^{-n} \\
&\Rightarrow \ f^{(0)}(1) = 0, \ f^{(1)}(1) = 1, \ f^{(n)}(1) = (-1)^{n-1}(n-1)!
\end{aligned}
$$

So the Taylor series of f about 1 is

$$\sum_{n=0}^{\infty} c_n(x - 1)^n, \text{ where } c_n = \frac{f^n(1)}{n!}.$$

$$
\begin{aligned}
\Rightarrow \ c_0 = 0, \ c_n &= \frac{(-1)^{n-1}(n-1)!}{n!} \quad n \geq 1 \\
&= \frac{(-1)^{n-1}}{n}.
\end{aligned}
$$

So the Taylor series of $\ln x$ is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x - 1)^n \qquad \circledast$$

Radius of convergence:

$$\begin{aligned}
\Lambda &= \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty}\frac{1}{n+1}\cdot\frac{n}{1} \\
&= \lim_{n\to\infty}\frac{n}{n+1} = 1 \\
\Rightarrow r &= \frac{1}{\Lambda} = 1, \text{ so the series converges for } |x-1| < 1.
\end{aligned}$$

Remarks:

1. This holds for $0 < x < 2$ by what we have shown: in fact it is also ok for $x = 2$.

2. We can show that $\circledast$ does infact converge to $\ln x$, i.e.

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n \quad \text{for } 1 < x \le 2$$

3. Plugging in $x = 2 \Rightarrow \ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}+...$ (alternating harmonic series).

18.5.2 Example

Find the Taylor series of $f(x) = e^x$ about $x = a$. Determine its radius of convergence.

$$f(x) \quad \Rightarrow \quad f^{(n)}(x) = e^x \text{ for all } n \geq 0$$
$$\Rightarrow \quad f^{(n)}(a) = e^a \text{ for all } n \geq 0$$
$$\Rightarrow \quad \text{the Taylor series of } f \text{ about } x = a \text{ is}$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \;=\; \sum_{n=0}^{\infty} \frac{e^a}{n!}(x-a)^n$$
$$=\; \sum_{n=0}^{\infty} c_n(x-a)^n$$

Radius of convergence:

$$\Lambda \;=\; \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty}\left|\frac{e^a}{(n+1)!}\cdot\frac{n!}{e^a}\right|$$
$$=\; \lim_{n\to\infty}\frac{1}{n+1} = 0$$

$$\Lambda \;=\; 0 \;\Rightarrow\; r = \infty.$$

Note:

1. Here, too, the Taylor series actually converges to the function, i.e.
$$e^x = \sum_{n=0}^{\infty} \frac{e^a}{n!}(x-a)^n.$$

2. Taking $a = 0$ gives the MacLaurin series: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

18.5.3   Example

Find the MacLaurin series for $f(x) = \sin x$. Determine its radius of convergence.

$$
\begin{aligned}
f(x) &= \sin x \;\Rightarrow\; f'(x) = \cos x, \; f''(x) = -\sin x, \; f'''(x) = -\cos x, \\
f^{(4)}(x) &= \sin x \text{ etc.}
\end{aligned}
$$

$$
\Rightarrow \quad f(0) = 0, \; f'(0) = 1, \; f''(0) = 0, \; f'''(0) = -1, \; f^{(4)}(0) = (0) \text{ etc.}
$$

$$
\text{So } f^{(n)}(0) = \begin{cases} 0 & n = 2m \\ (-1)^m & n = 2m + 1 \end{cases}
$$

$\Rightarrow$ MacLaurin series for $\sin x$ is:

$$
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ....
$$

$$
= \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!} \quad \circledast
$$

To determine the radius of convergence we use the ratio test on $\circledast$. So

$$
\begin{aligned}
L &= \lim_{m \to \infty} \left| \frac{a_{m+1}}{a_m} \right| \\
&= \lim_{m \to \infty} \left| \frac{(-1)^{m+1} x^{2m+3}}{(2m+3)!} \cdot \frac{(2m+1)!}{(-1)^m x^{2m+1}} \right| \\
&= \lim_{m \to \infty} \left| \frac{x^2}{(2m+3)(2m+2)} \right| \\
&= 0 < 1 \Rightarrow \text{ series converges for all } x \text{ (i.e. } r = \infty).
\end{aligned}
$$

Remarks :

1. The MacLaurin series converges to $\sin x$.

2. The fith-order Taylor polynomial is $T_5(x) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}$: Suppose now $0 \le x \le \frac{\pi}{4}$: the the remainder R_5 satisfies

$$
\begin{aligned}
|f(x) - R_5(x)| = |T_5(x)| &= \frac{|f^{(6)}(z)|}{6!} x^6, \text{ for some } z, \; 0 < z < x. \\
&= \frac{|\sin(z)|}{6!} x^6 \le \frac{1}{\sqrt{2}} \cdot \frac{1}{6!} (\frac{\pi}{4})^6 = 0.0023
\end{aligned}
$$

NOTE: In fact $T_5 = T_6$ for this function, so we can improve the error estimate by using R_6 instead of R_5 to 0.000026.

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18.5.4 Example

Use the above example of $\sin x$ to find the MacLaurin series for $\dfrac{\sin x}{x}$. Verify that $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$.

From above have

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \ , \ \text{converges absolutely for all } x$$

For $x \neq 0$:

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots$$

so

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots \right) = 1.$$

18.6 Binomial series

For a non-negative integer m, the binomial series is

$$(1 + x)^m = \sum_{n=0}^{m} \binom{m}{n} x^n,$$

where $\binom{m}{n}$ are the binomial coefficients defined by

$$\binom{m}{0} = 1, \qquad \binom{m}{n} = \frac{m(m-1)\ldots(m-n+1)}{n!} \quad \text{for} \quad n = 1, 2, \ldots$$

Note that in this expression for the binomial coefficient, m can be any real number. In the case that m is a positive integer and $0 \leq n \leq m$, such as in the above expansion, it can be written in the more familiar form

$$\binom{m}{n} = \frac{m!}{(m-n)!n!}.$$

Also note that when both m and n are positive integers, with $n \geq m + 1$ the binomial coefficient is zero. The above expansion can then be written

$$(1 + x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n.$$

Generalize the binomial theorem to arbitrary real powers by finding the MacLaurin series expansion of the function $f(x) = (1+x)^\alpha$ where $\alpha \in \mathbb{R}$. Find also the radius of convergence of the series.

$$f(x) = (1+x)^\alpha \Rightarrow f'(x) = \alpha(1+x)^{\alpha-1}$$
$$f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}$$

and, in general,

$$f^{(n)}(x) = \alpha(\alpha-1)\cdots\cdots(\alpha-n+1)(1+x)^{\alpha-n}$$
$$\Rightarrow \quad f(0) = 1, \quad f^{(n)}(0) = \alpha(\alpha-1)\cdots\cdots(\alpha-n+1), \text{ for } n \geq 1.$$

In terms of the binomial coefficient, we have

$$f^{(n)}(0) = n! \binom{\alpha}{n}, \qquad \text{for} \qquad n = 0, 1, 2, \ldots$$

Therefore the MacLaurin series expansion for $f(x)$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n.$$

Radius of convergence:

If $\alpha \in \mathbb{N}$, the series terminates after $\alpha + 1$ terms. A series with a finite number of terms is convergent for all real x. Note that in this case $\frac{c_n}{c_{n+1}}$ is not defined (division by 0).
For $\alpha \notin \mathbb{N}$ radius of convergence is

$$r = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!} \cdot \frac{(n+1)!}{\alpha(\alpha-1)\ldots(\alpha-n)} \right|$$
$$= \lim_{n\to\infty} \left| \frac{n+1}{\alpha-n} \right| = \lim_{n\to\infty} \left| \frac{1+\frac{1}{n}}{\frac{\alpha}{n}-1} \right| = 1.$$

Hence the series converges for $|x| < 1$.

18.6.1 Relationship to geometric series

A familiar series is the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots, \qquad \text{for} \qquad |x| < 1.$$

This result follows from the binomial theorem. Taking $\alpha = -1$ and replacing x with $-x$ we have

$$f(x) = (1-x)^{-1} = \frac{1}{1-x}.$$

The binomial coefficients in this case are

$$\binom{-1}{n} = \frac{(-1)(-2)\ldots(-n)}{n!} = (-1)^n, \qquad \text{for} \qquad n \geq 1.$$

The binomial theorem then gives, for $|x| < 1$,

$$\frac{1}{1-x} = 1 + \sum_{n=1}^{\infty}(-1)^n(-x)^n = \sum_{n=0}^{\infty} x^n,$$

which is the geometric series.

Notes

19 Appendix - Practice Problems

You should attempt the practice problems in this section while learning the material in the course. Try to solve all questions. Do at least questions marked with a star (*) before attempting assignment questions. Have a look at the questions before your tutorial. Try to solve them yourself first, and ask your tutor at the tutorial if you need help. Solutions to questions on this sheet are not to be handed in.

19.1 Numbers

1. Show that $\sqrt{2}$ is irrational.

2. Solve for x.

   (a) $|x + 1| < 1$
   (b) $|x + 1| > 1$
   (c) $|3x + 5| \leq 2$

3. Simplify.

   (a) $\frac{3+i}{3-i}$
   (b) $\frac{(2+3i)(3+2i)}{1+i}$

4. Write $z = 1 - \sqrt{3}i$ in polar form.

* 5. Find the roots of $x^2 - 4x + 5 = 0$.

19.2 Functions

1. Graph the following functions and give their domain and range.

   (a) $f(x) = x^2 + 2$
   *(b) $f(x) = \frac{1}{x-2}$
   *(c) $f(x) = -\frac{1}{x-2}$
   (d) $f(x) = \frac{1}{x^2+2}$
   *(e) $f(x) = \frac{x+2}{x^2-4}$
   *(f) $f(x) = \sqrt{x^2 - 9}$
   (g) $f(x) = -\sqrt{x^2 + 9}$
   (h) $f(x) = e^{x+1}$
   (i) $f(x) = 1 + e^{2x+3}$.

2. Stewart ed 5, p.56 question 2

3. Stewart ed 5, p.23 question 36

4. Stewart ed 5, p.23 question 38

5. Two species of kangaroos inhabit a national park. There are currently 4,500 grey kangaroos, whose population increases at a rate of 4% a year. There are only 3,000 red kangaroos, but they increase at a rate of 6% a year. In how many years will the red kangaroos outnumber the grey kangaroos? Use logarithms to find the answer and draw a graph of the two populations.

6. Let $P(t)$ be the amount of a radioactive material at time t. Experiments indicate $P(t)$ decays exponentially with time, so

$$P(t) = P_0 e^{-kt}$$

where P_0 is the amount of material at time $t = 0$ (initial amount) and k is a positive constant (called the decay constant). The half-life T is the time taken for the material to decay to one half of its initial amount. Obtain T in terms of k.

7. If $y = \tan(x)$, $0 \le x < \pi/2$, find $\cos(x)$ as a function of y.

* 8. Find the inverse of $f(x) = \frac{1}{3}\cos(2x)$ and state its domain and range.

9. For $f(x) = 3x + 2$, $g(x) = \ln(x^2 + 1)$ and $h(x) = 3e^{2x+1}$, find

   (a) $f(g(x))$
   (b) $g(f(x))$
   (c) $f(h(x))$
   (d) $h(f(x))$
   (e) $g(h(x))$
   (f) $f(g(h(x)))$
   (g) $h(g(f(x)))$.

10. Show from the definition that the following two functions are 1-1 on their domains (a graph is not sufficient here). Find the inverse function, and state the domain and range of both f and f^{-1}.

   (a) $f(x) = e^{3x+1}$
   * (b) $f(x) = \frac{x+1}{x+2}$.

19.3   Limits

1. Let
$$f(x) = \frac{x-3}{x^2-9}$$

   (a) Use the limit laws to find $L = \lim_{x \to 3} f(x)$.

   (b) Find a small interval around 3 where $|f(x) - L| \le .01$

* 2. Prove that $\lim_{x \to 0} x^4 \cos \frac{2x+1}{3x^2+5} = 0$.

3. Evaluate the following limits

   * (a) $\lim_{x \to 2} \frac{x^2+2x-8}{x^2-5x+6}$

   (b) $\lim_{x \to -1} \frac{x^2-x-12}{x+3}$

   (c) $\lim_{x \to -1} \frac{x+1}{x^2-x-2}$

   (d) $\lim_{t \to 1} \frac{t^3-t}{t^2-1}$

   (e) $\lim_{x \to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x}$ (Hint: multiply top and bottom by the conjugate surd $\sqrt{x+2} + \sqrt{2}$).

4. Show
$$\lim_{x \to 3} \frac{x}{5} = \frac{3}{5}$$
   using the ϵ, δ definition of the limit.

5. Prove that
$$\lim_{x \to 0} x \sin(\frac{1}{x}) = 0.$$

6. Evaluate the following limits.

   (a) $\lim_{x \to \infty} \frac{3x^2+2x+1}{x^2+x+3}$

   * (b) $\lim_{x \to \infty} (\sqrt{x+1} - \sqrt{x})$ (Hint: Use the method of Question 9(e) on Practice Sheet 2)

   (c) $\lim_{x \to \infty} \frac{x^2-1}{x^3+2}$

   (d) $\lim_{x \to \infty} \frac{x-x^3}{x^2+1}$

   (e) $\lim_{x \to -\infty} \frac{x+1}{1-x}$.

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19.4 Continuity

1. Determine whether or not $f(x)$ is continuous at $x = 1$ and sketch the graph of $f(x)$ for the following choices of $f(x)$:

   (a) $f(x) = \begin{cases} x^2 - 2x + 2 & , \quad \text{if } x < 1 \\ 3 - x & , \quad \text{if } x \geq 1 \end{cases}$

   (b) $f(x) = \begin{cases} x^3 & , \quad \text{if } x < 1 \\ (x - 2)^2 & , \quad \text{if } x \geq 1 \end{cases}$

   (c) $f(x) = \begin{cases} \frac{x-1}{x^2-1} & , \quad \text{if } x \neq 1 \\ 6 & , \quad \text{if } x = 1. \end{cases}$

* 2. For what value of a is $f(x)$ continuous for all x, where

$$f(x) = \begin{cases} \frac{x^2 - 3x + 2}{x - 1}, & x \neq 1 \\ a, & x = 1 \end{cases}$$

Give clear explanations.

3. Is the following function continuous at $x = -5$?

$$f(x) = \begin{cases} \frac{x+5}{x^2-25}, & x \neq -5 \\ -\frac{1}{10}, & x = -5 \end{cases}$$

19.5 Derivatives

1. (a) Sketch the graph of $f(x) = x|x|$.

   (b) For which values of x is f differentiable?

   (c) Obtain a formula for $f'(x)$.

2. The hyperbolic sine and cosine functions are defined by

$$\sinh x = \frac{1}{2}(e^x - e^{-x}), \quad \cosh x = \frac{1}{2}(e^x + e^{-x})$$

respectively.

   (a) Prove $\cosh^2 x - \sinh^2 x = 1$.

   (b) Prove $\frac{d}{dx} \sinh x = \cosh x$, $\frac{d}{dx} \cosh x = \sinh x$.

   (c) Find $\lim\limits_{x \to 0} \frac{\sinh x}{x}$.

3. The Tasmanian Devil often preys on animals larger than itself. Researchers have identified the mathematical relationship

$$L(w) = 2.265 w^{2.543},$$

where w (in kg) is the weight and $L(w)$ (in mm) is the length of the Tasmanian Devil. Suppose that the weight of a particular Tasmanian Devil can be estimated by the function

$$w(t) = 0.125 + 0.18t,$$

where $w(t)$ is the weight (in kg) and t is the age, in weeks, of a Tasmanian Devil that is less than one year old. How fast is the length of a 30-week-old Tasmanian Devil changing?

4. Show that the equation of the tangent line to a point (x_0, y_0) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is given by

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1.$$

* 5. Consider the function $f(x) = x^x$. Evaluate $f'(x)$ for $x > 0$.

6. A new car costing P dollars depreciates at an annual rate of $r\%$, so that after t years the car will be worth C dollars where

$$C(t) = P(1 - \frac{r}{100})^t.$$

   (a) Find $\frac{dC}{dt}$

   (b) How much will the car be worth in the long run?

* 7. Find

$$\frac{d}{d\theta} \frac{\theta}{\cos(\sin \theta)}.$$

8. A spherical snowball is melting in such a way that its volume is decreasing at a rate of 1 cm^3/min. At what rate is the diameter decreasing when the diameter is 10 cm?

9. Using L'Hôpital's rule evaluate the following limits

   *(a) $\lim\limits_{\theta \to 0} \frac{1 - \cos \theta}{\theta^2}$

   (b) $\lim\limits_{x \to 0} \frac{e^x - 1}{\sin x}$

   (c) $\lim\limits_{x \to 0} \frac{\sin x - x}{x^3}$

   *(d) $\lim\limits_{x \to 1}\left(\frac{1}{\ln x} - \frac{1}{x - 1}\right)$

   (e) $\lim\limits_{x \to \infty} e^{-x} \ln x$

   (f) $\lim\limits_{\theta \to 0} \frac{(\sin \theta)^n}{\sin(\theta^n)}, \quad 1 \leq n \in \mathbb{N}.$

10. Use the mean value theorem to prove the following (this is the second derivative test).

    Suppose $f : [a, b] \to \mathbb{R}$ is continuous, and has continuous first and second derivatives on $[a, b]$. Then if $c \in (a, b)$ is a point with $f'(c) = 0$ and $f''(c) > 0$ then f has a local minimum at c.

11. Determine where the following functions are increasing/decreasing.

    (a) $f(x) = (x - 1)(x + 2) = x^2 + x - 2$

    (b) $f(x) = x|x|$.

    (c) $f(x) = \sin x, \ x \in (0, \pi)$

    (d) $f(x) = \frac{e^{-x} + 1}{e^x}$

12. Prove the following:

$$e^x \geq x + 1, \text{ for all } x \geq 0.$$

    Hint: Consider $f(x) = e^x - x - 1$.

13. Prove the following:
$$\sin x \leq x, \text{ for all } x \geq 0.$$

14. Find the absolute maximum and minimum of the function $f(x) = x^3 - 3x + 2$ on the closed interval $[0, 2]$.

15. Find all relative maxima and minima and thus sketch the graph of the function
$$f(x) = x + \frac{1}{x}.$$

* 16. Show that the function

$$f(x) = \begin{cases} x \ln x & , \quad x > 0 \\ 0 & , \quad x = 0 \end{cases}$$

    is continuous on $[0, \infty)$. What are the absolute maximum and minimum values of $f(x)$ on $[0, 2]$?

17. To increase the velocity of the air flowing through the trachea when a human coughs, the body contracts the windpipe, producing a more effective cough. Tuchinsky formulated that the velocity of air that is flowing through the trachea during a cough is

$$V = C(R_0 - R)R^2,$$

    where $C > 0$ is a constant based on individual body characteristics, R_0 is the radius of the windpipe before the cough, and R is the radius of the windpipe during the cough. Find the value of R that maximizes the velocity. (Interestingly, Tuchinsky also states that X-rays indicate that the body naturally contracts the windpipe to this radius during a cough.)

19.6   Integration

1. Evaluate

   (a) $\int (x^3 - 2x + 1)dx$

   (b) $\int x^{-a}dx, \ a > 0$.

   (c) $\int \sin(x + \frac{\pi}{3} + 2x)dx$

   (d) $\int (x + \frac{1}{x})^2 dx$

2. Let f be continuous on an interval I. Let F_1 and F_2 be two different anti-derivatives of f. Show that F_1 and F_2 differ by a constant only. (Hint: Recall that if $f'(x) = 0$ on an interval I then f is constant on this interval.)

3. Estimate the area under the below function using:

   (a) the right rectangle rule (Riemann sums) with 2 subintervals, then 4 subintervals then 8 subintervals.

   (b) the trapezoidal rule with 4 subintervals.
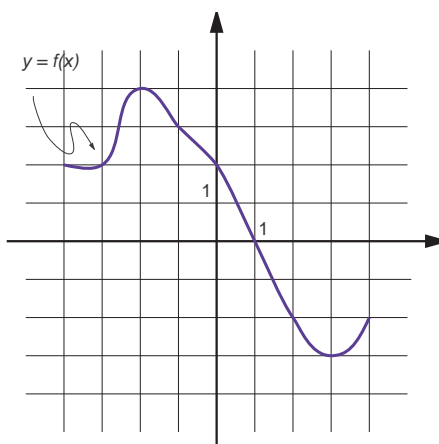
4. What is wrong with the following argument.

   $F(x) = -1/(x-1)$ is an anti derivative of the function $f(x) = 1/(x-1)^2$. Thus by the fundamental theorem, the area under f on the interval $[0, 3]$ is given by

   $$\int_0^3 \frac{1}{(x-1)^2} dx = \frac{-1}{(x-1)} \Big|_0^3 = \frac{-1}{2} - \frac{-1}{-1} = -\frac{3}{2}.$$

   Does this show a positive function can have a negative area?

* 5. Find the area enclosed by the graphs of the functions $f(x) = x^2$ and $g(x) = x$.

6. Derive a formula for the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

7. Find the area under the graph of

    (a) $y = \sqrt{4 - x^2}$ above $[0, 1]$
    (b) $y = \sec x \tan x$ above $[0, \frac{\pi}{3}]$.

8. Evaluate each of the following integrals by interpreting it in terms of areas:

    (a) $\int_1^3 (1+2x)dx$ (b) $\int_{-3}^0 (1+\sqrt{9-x^2})dx$ (c) $\int_{-2}^2 |x|dx$ (d) $\int_{-2}^2 \sqrt{4-x^2}dx$.

9. Determine $\int_0^\infty xe^{-x^2}dx$.

10. By finding a suitable substitution, evaluate the following indefinite integrals:

    * (a) $\int \frac{e^x}{1+e^{2x}}dx$
    (b) $\int \frac{dx}{x^2+4x+9}dx$
    (c) $\int \cos x \sin^3 xdx$.

11. Use integration by parts to find:

    * (a) $\int x \cos xdx$
    (b) $\int \cos^2 xdx$
    (c) $\int e^x \cos xdx$
    (d) $\int \ln xdx$ (e) $\int x \sec^2 xdx$

12. Use partial fractions to find

    * (a) $\int \frac{x}{(x+1)(x+2)}dx$
    (b) $\int \frac{x+3}{x(x+2)}dx$
    (c) $\int \frac{4x+8}{x^2+2x-3}dx$.

13. Evaluate the following integrals:

    (a) $\int \frac{x}{x^2-1}dx$ (b) $\int \frac{1+e^x}{1-e^x}dx$.

14. Evaluate the following indefinite integrals

    (a) $\int \frac{\sin(1/t^2)}{t^3} dt$
    (b) $\int \sin(x) \cos(\cos(x)) dx$

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Check by explicitly differentating your answer to confirm it is indeed an anti-derivative.

15. Find the volume of the ellipsoid obtained by rotating the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the x-axis.

16. Let $a > 0$ be constant. Find the volume V of the solid of revolution obtained by rotating $y = f(x)$ above the interval $[0, a]$ about the x-axis, for the following choices of $f(x)$:

    (a) $y = x^2$
  * (b) $y = \sqrt{\ln(x+1)}$.

17. Use Euler's formula to express $\cos(2\theta)$ and $\sin(2\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$.

19.7 Sequences

1. Use L'Hôpital's rule to find

$$\lim_{n\to\infty} \frac{\sin(1/n)}{\log(1+1/n)}.$$

2. Evaluate the limit of the sequence $\{a_n\}_{n=1}^{\infty}$ for the following choices of a_n:

    (a) $a_n = nc^n$, $0 \le c < 1$
  * (b) $a_n = \frac{1+2^n}{1+3^n}$.

* 3. Prove that
$$\lim_{n\to\infty} n^{1/n} = 1.$$

19.8 Series

* 1. It is known that a repeated decimal can always be expressed as a fraction. Consider, for example, the decimal $0.727272\ldots$. Use the fact that

$$0.727272\cdots = 0.72 + 0.0072 + 0.000072 + \ldots$$

to express the decimal as a geometric series. Hence show that

$$0.727272\cdots = \frac{8}{11}.$$

2. Determine whether or not the following series converge:

    (a) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

(b) $\frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \ldots$

3. (a) Prove that $\sum\limits_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$, for $|x| < 1$.

   (Hint: Consider $x\frac{d}{dx}(\frac{1}{1-x})$)

   (b) Hence, or otherwise, evaluate $\sum\limits_{n=1}^{\infty} \frac{n}{2^n}$.

4. Consider the $p = 1$ case of the p-series (called the harmonic series):

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}\right) + \ldots$$

   and group together its terms as shown. Show that the sum of each group of fractions is more than $\frac{1}{2}$. Hence deduce that the harmonic series diverges.

* 5. Evaluate the following series:

   (a) $\sum\limits_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$

   (b) $\sum\limits_{n=0}^{\infty} (-1)^n x^n$.

6. (a) For $p > 0$, show that $f(x) = x^p$ is strictly increasing on $[0, \infty)$.

   (b) For $0 < p < 1$, show that

   $$\frac{1}{n^p} \geq \frac{1}{n}, \quad n = 1, 2, 3 \ldots$$

   (c) Show that the p-series

   $$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

   diverges for $0 \leq p \leq 1$.

7. Determine whether or not the following series converge.

   (a) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n3^n}$

   * (b) $\sum\limits_{n=0}^{\infty} \frac{2^{2n}}{(2n)!}$

   * (c) $\sum\limits_{n=1}^{\infty} \frac{\ln(n)}{n!}$.

8. (a) Write down from the examples in lectures or the text book two series of positive terms $\sum a_n$ & $\sum b_n$ satisfying the following conditions:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1 \text{ and the } \sum_m a_n \text{ converges.}$$

$$\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = 1 \text{ and } \sum_n b_n \text{ diverges.}$$

This shows the ratio test gives no information if the limit is 1. You can accept any proofs of convergence in lectures or the text book but prove both the limits above.

9. Find an example of a sequence a_n with $a_n = f(n)/g(n)$ for differentiable functions f and g, where $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} f'(n)/g'(n)$ both exist, and

$$\lim_{n\to\infty} a_n \neq \lim_{n\to\infty} \frac{f'(n)}{g'(n)}.$$

This shows before using L'Hôpital's rule we must remember to check that either $\lim_{n\to\infty} f(n) = 0 = \lim_{n\to\infty} g(n)$ or $\lim_{n\to\infty} f(n) = \pm\infty = \lim_{n\to\infty} g(n)$!

19.9 Power series and Taylor series

1. This question is about the MacLaurin series for

$$(1 + x)^{3/2}.$$

Find the first three terms of this series. If this series is written as $\sum_{n=0}^{\infty} a_n$, find the formula for a_n. Find a formula for c_n so $a_{n+1} = c_n a_n$.

* 2. Obtain the MacLaurin series expansion for $\arctan x$ and its radius of convergence.
   (**Hint**: Use $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$).
   Using the fact that $\tan(\pi/6) = 1/\sqrt{3}$, show that

$$\pi = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}.$$

* 3. Obtain the MacLaurin series expansions and radius of convergence for the following functions:

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) \quad , \quad \sinh x = \frac{1}{2}(e^x - e^{-x}).$$

* 4. Show that for $|x| < 1$,

(a)
$$\sqrt{1+x} = 1 + \sum_{n=0}^{\infty} \frac{(-1)^{n-1}(2n-2)!}{2^{2n-1}n!(n-1)!}x^n.$$

(b)
$$\frac{1}{\sqrt{1+x}} = \sum_{n=0}^{\infty} \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}x^n.$$

5. The theory of relativity predicts that the mass m of an object when it is moving at a speed v is given by the formula

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where c is the speed of light and m_0 is the mass of the object when it is at rest. Express m as a MacLaurin series in $\frac{v}{c}$ and determine for which values of v this series converges. (**Hint**: Use a result from the previous question.)

6. Show that the functions

   (a) $f(x) = (1 + x^2)e^{-x^2}$
   (b) $f(x) = x^2 - \ln(1 + x^2)$

   both satisfy $f'(x) = f''(x) = 0$ at $x = 0$ (so the second derivative test fails here). Show, using Taylor series, that (a) has a relative maximum and (b) a relative minimum at $x = 0$.

* 7. Find the sum of each of the following convergent series by recognizing it as a Taylor series evaluated at a particular value of x.

   (a) $1 + \frac{2}{1!} + \frac{4}{2!} + \frac{8}{3!} + \cdots + \frac{2^n}{n!} + \ldots$
   (b) $1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots + \frac{(-1)^n}{(2n+1)!} + \ldots$
   (c) $1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots + \left(\frac{1}{4}\right)^n + \ldots$
   (d) $1 - \frac{100}{2!} + \frac{10000}{4!} + \cdots + \frac{(-1)^n \cdot 10^{2n}}{(2n)!} + \ldots$

19.10   Vectors

1. A canoe travels at a speed 3km/h in a direction due west across a river flowing due north at 4km/h. Find the actual speed and direction of the canoe.

2. Draw a clock with vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{12}$ pointing from the centre to each of $1, 2, \ldots, 12$. Describe each of the following vectors.

(a) $\boldsymbol{v}_1 + \cdots + \boldsymbol{v}_{12}$.

(b) $\boldsymbol{v}_1 + \boldsymbol{v}_2 + 3\boldsymbol{v}_3 + \boldsymbol{v}_4 + \boldsymbol{v}_5 + \boldsymbol{v}_6 + \boldsymbol{v}_7 + \boldsymbol{v}_8 + 3\boldsymbol{v}_9 + \boldsymbol{v}_{10} + \boldsymbol{v}_{11} + \boldsymbol{v}_{12}$.

(c) $\boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3 + \boldsymbol{v}_4 + \boldsymbol{v}_6 + \boldsymbol{v}_7 + \boldsymbol{v}_9 + \boldsymbol{v}_{10} + \boldsymbol{v}_{12}$.

3. Find the lengths of the sides of the triangle ABC and determine whether the triangle is a right triangle.

   (a) $A = (4, 2, 0)$, $B = (6, 6, 8)$, $C = (10, 8, 6)$

   (b) $A = (5, 5, 1)$, $B = (6, 3, 2)$, $C = (1, 4, 4)$.

4. (a) Suppose $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal vectors. Prove that $\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 = \|\boldsymbol{u} + \boldsymbol{v}\|^2$. What geometric fact have you proved?

   (b) Prove that $\|\boldsymbol{u} + \boldsymbol{v}\|^2 + \|\boldsymbol{u} - \boldsymbol{v}\|^2 = 2(\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2)$. What geometric fact have you proved?

   (c) Consider a triangle with side lengths a, b, c. Bisect side c, and draw a line connecting the midpoint to the opposite vertex. Let the length of this line be m. Using vectors prove that $a^2 + b^2 = 2(m^2 + (c/2)^2)$. This is called Apollonius' Theorem.

* 5. Find a non-zero vector in $\mathbb{R}^3$ orthogonal to $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$.

6. Given the point $P = (2, -1, 4)$, find a point Q where $\overrightarrow{PQ}$ has the same direction as $\boldsymbol{v} = 7\boldsymbol{i} + 6\boldsymbol{j} - 3\boldsymbol{k}$.

* 7. For what values of c is the angle between the vectors $(1, 2, 1)$ and $(1, 1, c)$ equal to $60°$? For what values of c are they orthogonal?

8. Determine whether or not each of the following sets of vectors are linearly independent.

   (a) $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$.

   (b) $\boldsymbol{i} + \boldsymbol{j}$, $\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{i} - \boldsymbol{j}$.

   (c) $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 4 \\ 1 \\ 0 \end{pmatrix}$.

9. Prove that the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

form a basis for $\mathbb{R}^3$. What are the components of the vector $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ in this basis?

10. Given

$$v_1 = \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix}, v_2 = \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix}, v_3 = \begin{pmatrix} 2 \\ -9 \\ h \end{pmatrix},$$

for what values of h is v_3 in $span\{v_1, v_2\}$, and for what values of h is $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer.

* 11. Which of the following subsets of $\mathbb{R}^3$ are vector spaces? Find a basis in each case.

   (a) $\{(a, b, a + 2b) \mid a, b \in \mathbb{R}\}$.
   (b) $\{(a, b, c) | a - b = 3c + 1\}$.

12. Suppose $S = \{v_1, v_2, \ldots, v_m\} \subset \mathbb{R}^n$ and let V be the set of all linear combinations of the elements of S, so

$$V = \{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m \mid \alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}\}.$$

   Prove that V is a vector space.

13. If $\{a, b, c\}$ forms a basis for a 3-dimensional vector space, could $\{a+b, b+c, c+a\}$ also be a basis? Give your reasons.

* 14. Let $u_1 = (0, 0, 3)$, $u_2 = (-2, -1, 0)$, $u_3 = (1, 0, 0)$. Which of u_1, u_2, u_3 are linear combinations of $v_1 = (1, 2, 3)$, $v_2 = (4, 5, 6)$, $v_3 = (7, 8, 9)$? Do $\{v_1, v_2, v_3\}$ form a basis of $\mathbb{R}^3$?

15. Determine if the following vectors are a basis for $\mathbb{R}^3$. Justify your answer.

$$\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 5 \\ -7 \\ 4 \end{pmatrix}, \begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix}$$

16. Let $v_1 = (1, 2, 3)$, $v_2 = (4, 5, 6)$, $v_3 = (-2, -1, 0)$. Do $\{v_1, v_2, v_3\}$ form a basis of $\mathbb{R}^3$? Explain.

* 17. Let $v_1 = (1, 4, 6)$, $v_2 = (2, 0, 1)$, $v_3 = (-1, 12, 16)$. Let $V = span\{v_1, v_2, v_3\}$. Find $\dim V$.

19.11   Matrix Algebra

1. Let $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Find simple expressions for A^2 and A^3. What do you notice?

2. Find 2×2 matrices A, B such that $(A+B)^2 \neq A^2 + 2AB + B^2$.

* 3. Prove that $(A+B)^T = A^T + B^T$ for any $m \times n$ matrices A and B.

4. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Find all 2×2 matrices B such that $AB = BA$.

5. Find a 2×2 matrix A with no entry of A equal to 0 such that $A^2 = 0$.

6. A matrix A is called *idempotent* if $A^2 = A$.

   (a) Find x so that A is idempotent:

$$A = \begin{pmatrix} 2/3 & 0 & x \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{3} & 0 & 1/3 \end{pmatrix}$$

   (b) What is the smallest (integer) index N so that $B^N = 0$? A matrix with such a property is called *nilpotent*.

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

7. Let $C = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 3 & -1 \end{pmatrix}$. Show that C is invertible, ie show linear independence of a certain set of vectors.

* 8. Let $A = \begin{pmatrix} 4 & -1 & 2 \\ 3 & 0 & x \\ 8 & -3 & 2 \end{pmatrix}$. For which value(s) of x is A *not* invertible?

9. Find the inverse matrix for:

   (a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

* 10. Let

$$A = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{1}{5} & \frac{2}{5} \\ 0 & 0 & \frac{1}{5} \end{pmatrix} \text{ and } B = \begin{pmatrix} -1 & 2 & x \\ 3 & -1 & y \\ 0 & 0 & z \end{pmatrix}.$$

Find real numbers x, y and z such that $B = A^{-1}$.

11. A swimming pool can be filled by 3 pipes A,B and C. Pipe A can fill the pool by itself in 8 hours. Pipes A and C used together can fill the pool in 6 hours and pipes B and C used together can fill the pool in 10 hours. How long does it take to fill the pool if all 3 pipes are used?

* 12. Use Gaussian elimination to solve the following system of equations, and check your answer by substitution.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= -6 \\ 4x_1 + 2x_3 &= 2 \\ 3x_1 + x_2 + 2x_3 &= 3 \end{aligned}$$

13. Let $\boldsymbol{x}_p$ be a particular solution of $A\boldsymbol{x} = \boldsymbol{b}$. Show that every solution of this equation is of the form $\boldsymbol{x} = \boldsymbol{x}_p + \mathbf{h}$ where $\mathbf{h}$ is a solution of the homogeneous equation $A\boldsymbol{x} = \mathbf{0}$.

14. Show that the set of solutions of

$$A\boldsymbol{x} = \mathbf{b}, \text{ with } \mathbf{b} = \mathbf{0}$$

form a vector space for A.
If $\mathbf{b} \neq \mathbf{0}$, does the set of solutions still form a vector space?

15. Show that $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ satisfies $A^2 - 4A + 3I = 0$. Hence deduce that A is invertible with inverse $A^{-1} = \frac{1}{3}(4I - A)$.

16. Let $a \in \mathbb{R}$. Use Gauss-Jordan elimination to find the inverses of the following matrices. Check by multiplication if your inverse is correct.

* (a) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & \alpha \\ 2 & 1 & 1-\alpha \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 0 & -1 \\ 1 & 2 & 1 \\ 1 & 0 & \alpha \end{pmatrix}.$

17. Find the inverses of the following matrices.

(a) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$.

(b) $\begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}$.

(c) $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

18. Let $A = \begin{pmatrix} 1 & 0 & 5 \\ 2 & \text{-}1 & 2 \\ 3 & 1 & 2 \end{pmatrix}$. Calculate the determinant of A.

19. Determine for which values of x the following determinants vanish:

(a) $\begin{vmatrix} 1-x & 2 \\ 3 & 2-x \end{vmatrix}$

(b) $\begin{vmatrix} 1 & 2 \\ x^2 & x \end{vmatrix}$

* (c) $\begin{vmatrix} 1 & 1 & 4 \\ 2 & x & 1 \\ 2x & x^2 & x \end{vmatrix}$.

20. Evaluate the following determinants

(a) $\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$

(b) $\begin{vmatrix} 3 & 1 & 0 \\ 0 & 6 & 4 \\ 0 & 0 & 1 \end{vmatrix}$

(c) $\begin{vmatrix} -1 & 1 & 3 & 4 \\ 3 & 2 & 4 & -1 \\ 0 & 3 & 1 & 0 \\ 4 & 1 & -1 & -5 \end{vmatrix}$

(d) $\begin{vmatrix} -1 & 1 & 2 \\ 2 & 1 & -1 \\ -1 & 1 & -1 \end{vmatrix}$

* 21. Use determinants to establish the linear dependence or independence of the following row vectors:

(a) $(4, 1, 4), (2, 0, 4), (-1, 2, 2)$

(b) $(3, -3, -1), (6, -6, 0), (3, -3, 2)$

(c) $(1, 1, 1), (0, 1, 0), (0, 1, -1)$

* 22. For an invertible matrix A, prove that A^n is also invertible, for all natural numbers n.

20   Appendix - Mathematical notation

$>$	is strictly greater than	e.g. $5 > 2$
$<$	is strictly less than	e.g. $2 < 5$
$\geq, \leq$	is greater/less than or equal to	e.g. $2 \leq 2, 4 \geq 3$
$\in$	is an element of	e.g. $3.5 \in \mathbb{R}$
$\notin$	is not an element of	e.g. $3.5 \notin \mathbb{N}$
$\subseteq$	is a subset of	e.g. $\mathbb{N} \subset \mathbb{R}$
$\{x \vert ...\}$	set of all x for which a condition is fulfilled	e.g. $\{x \in \mathbb{R} \vert x > 0\}$
$\forall$	for all	e.g. $\forall x \in \mathbb{R}$
...	continue the pattern	e.g. $1, 2, 3, ..., 9, 10$
$n!$	n factorial, $n! = n \cdot (n-1) \cdot ... \cdot 3 \cdot 2 \cdot 1$	e.g. $3! = 3 \cdot 2 \cdot 1 = 6$
$\Rightarrow$	implies, if ... then	e.g. $x = 2 \Rightarrow x^2 = 4$
$\Leftrightarrow$	is equivalent to, if and only if	e.g. $x^2 = 4 \Leftrightarrow x = 2$ or $x = -2$

21 Appendix - Basics

This appendix contains some very important basic mathematical rules. Since this is highschool material, you must know and be able to apply all of them to successfully complete MATH1051. Go through the sections and check your knowledge - do some additional exercises from textbooks if you encounter problems, or check the MATH1040/MATH1050 material on the course websites.

21.1 Powers

21.1.1 Notation (product of two real numbers)

The product of two real numbers a and b can be written in either of the following ways:
$$a \times b = a \cdot b = ab.$$

21.1.2 Notation (powers)

The product after multiplying the same real number a n-times is

$$\underbrace{aa \cdots a}_{n \text{ factors}} = a^n.$$

Note that $a^n = aa^{n-1} = a^{n-1}a$.
Then $a^0 = 1$, $a^1 = a$, $aa = a^2 = aa^1$, $aaa = a^3 = aa^2$.
We also have $0^0 = 1$ and $0^n = 0$, for $n \neq 0$.

21.1.3 Power rules

Let $a, b, c \in \mathbb{R}$. We set
$$a^{-b} = \frac{1}{a^b}.$$

Then
$$a^b a^c = a^{b+c}, \quad a > 0 \text{ and}$$
$$\left(a^b\right)^c = a^{bc}.$$

We also have
$$a^{\frac{p}{q}} = \sqrt[q]{a^p},$$

with $p, q \in \mathbb{Z}$ and $q > 0$.
These formulae also apply for some other values of a, p and q but care needs to be taken.
For example, $a = -1$, $p = 1$. If $q = 3$, then $(-1)^{\frac{1}{3}} = -1$.
But if $a = 2$, then $(-1)^{\frac{1}{2}} = i$, which is not a real but a complex number!

21.1.4   Examples

1. $(a \times a) \times (a \times a) \times (a \times a) = \left(a^2\right)^3 = a^{2 \times 3} = a^6$

2. $a^2 \times a^3 = a^{2+3} = a^5$

3. $a^{\frac{1}{2}} = \sqrt{a}$

4. $a^{\frac{1}{3}} = \sqrt[3]{a}$

21.2   Multiplication/Addition of real numbers

21.2.1   Laws (multiplication and addition)

1. $a + b = b + a$ and $ab = ba$ (Commutative Law)

2. $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$ (Associative Law)

3. $a(c + d) = ac + ad = (c + d)a = ca + da$ (Distributive Law)

21.2.2   Examples (multiplications of sums)

1. $\mathbf{(a + b)(c + d) = ac + ad + bc + bd}$,

    as $(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd$.

2. $\mathbf{(a + b)^2 = a^2 + 2ab + b^2}$

    as $(a + b)^2 = (a + b)(a + b) = a(a + b) + b(a + b)$
    $= aa + ab + ba + bb = a^2 + 2ab + b^2$.

3. $\mathbf{(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3}$

    as $(a + b)^3 = (a + b)(a + b)^2 = (a + b)(a^2 + 2ab + b^2)$
    $= a(a^2 + 2ab + b^2) + b(a^2 + 2ab + b^2)$
    $= a^3 + 3a^2b + 3ab^2 + b^3$.

4. $\mathbf{(a - b)^2 = a^2 - 2ab + b^2}$

5. $\mathbf{(a + b)(a - b) = a^2 - b^2}$

21.3   Fractions

21.3.1   Rule (multiplying and dividing two fractions)

Let $a, b, c, d \in \mathbb{Z}$. Then
$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$
and
$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.$$

21.3.2   Rule (adding two fractions)

Let $a, b, c, d \in \mathbb{Z}$. Then
$$\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}.$$
If the two fractions have different denominator, a common denominator needs to be found:
$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

21.3.3   Examples

1. $\dfrac{x^2 + 1}{x} = \dfrac{x^2}{x} + \dfrac{1}{x} = x + \dfrac{1}{x}.$

2. $\dfrac{x}{x + 1} = \dfrac{(x + 1) - 1}{x + 1} = \dfrac{x + 1}{x + 1} - \dfrac{1}{x + 1} = 1 - \dfrac{1}{x + 1}$

3. Simplify
$$\frac{1}{x}\left(\frac{1}{x + 1} - \frac{2}{x + 2}\right)$$

Solution: First $\dfrac{1}{x + 1} - \dfrac{2}{x + 2} = \dfrac{1(x + 2) - 2(x + 1)}{(x + 1)(x + 2)} = \dfrac{-x}{(x + 1)(x + 2)}$

$\Rightarrow \dfrac{1}{x}\left(\dfrac{1}{x + 1} - \dfrac{2}{x + 2}\right) = \dfrac{1}{x}\left(\dfrac{-x}{(x + 1)(x + 2)}\right) = \dfrac{-1}{(x + 1)(x + 2)}$

21.4   Solving quadratic equations

A quadratic equation is an equation of the form $ax^2 + bx + c = 0$, or $x^2 + bx + c = 0$ if all coefficients are divided by a. Here a, b and c are given. Quite often you will be required to solve equations of this type, which means we try to determine x. There are many different ways of finding the solution. A few are listed here. They are completing the square, using a formula, or factorizing.

21.4.1 Completing the square

We may solve $x^2 + bx + c = 0$ by first writing the quadratic polynomial $x^2 + bx + c$ in the form $(x + d)^2 + e$, and then solving $(x + d)^2 + e = 0$.

Applying the formula from example 21.2.2.(2),

$x^2 + bx = (x + \frac{b}{2})^2 - (\frac{b}{2})^2$ (check!)
$\Rightarrow x^2 + bx + c = (x + \frac{b}{2})^2 - (\frac{b}{2})^2 + c = (x + d)^2 + e$, with $d = \frac{b}{2}$ and
$e = c - (\frac{b}{2})^2 = c - d^2$.

To solve: $(x + d)^2 + e = 0 \Leftrightarrow (x + d)^2 = -e \Leftrightarrow x + d = \pm\sqrt{-e} \Leftrightarrow x = -d - \sqrt{-e}$
or $x = -d + \sqrt{-e}$.

21.4.2 Examples

1. Express $x^2 + 6x + 5 = (x + d)^2 + e$. Then solve the equation $x^2 + 6x + 5 = 0$.

   $x^2 + 6x + 5 = (x + 3)^2 - 3^2 + 5 = (x + 3)^2 - 4$.
   We solve $x^2 + 6x + 5 = (x + 3)^2 - 4 = 0$
   $\Leftrightarrow (x + 3)^2 = 4 \Leftrightarrow x + 3 = \pm 2 \Leftrightarrow x = -5$ or $x = -1$.

2. The method of completing the square can also be used to find the turning point of a parabola.
   Express $f(x) = x^2 + 4x + 7 = (x + d)^2 + e$. Find the turning point of this parabola.

   Now $b = 4$ so $d = 2$.
   Also $c = 7$ so $e = 7 - 4 = 3$. Thus
   $x^2 + 4x + 7 = (x + 2)^2 + 3$.
   The turning point of a parabola $f(x) = (x + d)^2 + e$ is at $(-d, e)$.
   In this case the turning point is a minimum located at $(-2, 3)$.

21.4.3 Formula (Solution to quadratic equation)

There is also a formula you can use to solve a quadratic equation.
Let $a, b, c \in \mathbb{R}$.
The solution of the equation

$$ax^2 + bx + c = 0$$

is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The solution of the equation

$$x^2 + bx + c = 0$$

is

$$x = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - c}.$$

21.4.4 Factorizing

Another way of solving quadratic (and higher order polynomial) equations is by factorizing. We try to write the polynomial as a product of sums. You can sometimes use the formulas from Examples 21.2.2.(2),(4) or (5) for this. Factorizing is also used to simplify expressions.

If you cannot use one of the formulas from Examples 21.2.2.(2),(4) or (5), you can sometimes factorize the following way. We would like to express $x^2 + bx + c = (x + p)(x + q)$. Multiplying out we have $x^2 + bx + c = (x + p)(x + q) = x^2 + (p + q) + pq$. We now need to find p and q such that $b = p + q$ and $c = pq$. Have a look at these examples.

21.4.5 Examples

1. Factorize $f(x) = x^2 + 6x + 5$ using the formula from Example 21.2.2.(2). Then find the zeros.

   $x^2 + 6x + 5 = (x + 3)^2 - 2^2 = ([x + 3] - 2)([x + 3] + 2) = (x + 1)(x + 5)$.
   The zeros can be read directly from this expression:
   $(x + 1)(x + 5) = 0 \Rightarrow x = -1$ or $x = -5$.

2. Factorize $x^2 + 3x + 2$ without using the formula from Example 21.2.2.(2).

   $x^2 + 3x + 2 = x^2 + bx + c$. We must find p and q such that $b = 3 = p + q$ and $c = 2 = pq$. A solution is $p = 1$ and $q = 2$, so $x^2 + 3x + 2 = (x + 1)(x + 2)$.

3. Factorize $x^2 - 7x + 6$.

Now $-7 = p + q$ and $6 = pq$. A solution is $p = -6$ and $q = -1$, so $x^2 - 7x + 6 = (x - 6)(x - 1)$.

21.4.6 Factorizing using common factors

The following formula uses common factors to factorize :

$$a(\mathbf{x} + \mathbf{d}) + b(\mathbf{x} + \mathbf{d}) = (a + b)(\mathbf{x} + \mathbf{d}).$$

21.4.7 Examples

1. Factorize $x(\mathbf{x} + \mathbf{3}) + 2(\mathbf{x} + \mathbf{3})$.
   $x(x + 3) + 2(x + 3) = (x + 2)(x + 3)$
   setting $a = x$, $b = 2$, and $d = 3$ in the above formula.

2. Factorize $(x + 4)(x + 3) + 2(x + 3)$.
   $(x + 4)(x + 3) + 2(x + 3) = ([x + 4] + 2)(x + 3)$
   setting $a = [x + 4]$, $b = 2$, and $d = 3$.

21.5 Surds

The following formula is often used to "simplify" surds:

$$\sqrt{a} - \sqrt{b} = \frac{a - b}{\sqrt{a} + \sqrt{b}}.$$

To see this

$$
\begin{aligned}
\sqrt{a} - \sqrt{b} &= \left(\sqrt{a} - \sqrt{b}\right) \frac{\sqrt{a} + \sqrt{b}}{\sqrt{a} + \sqrt{b}} \\
&= \frac{(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})}{\sqrt{a} + \sqrt{b}} \\
&= \frac{(\sqrt{a})^2 - (\sqrt{b})^2}{\sqrt{a} + \sqrt{b}} = \frac{a - b}{\sqrt{a} + \sqrt{b}}.
\end{aligned}
$$

Similarly,

$$\sqrt{a} + \sqrt{b} = \frac{a - b}{\sqrt{a} - \sqrt{b}}$$

21.5.1   Example

Simplify

$$\frac{\sqrt{2x+1}-\sqrt{x+1}}{x}, \text{ where } x \neq 0$$

First

$$
\begin{aligned}
\sqrt{2x+1}-\sqrt{x+1} &= \frac{\left(\sqrt{2x+1}\right)^2 - \left(\sqrt{x+1}\right)^2}{\sqrt{2x+1}+\sqrt{x+1}} \\
&= \frac{x}{\sqrt{2x+1}+\sqrt{x+1}} \\
\Rightarrow \frac{\sqrt{2x+1}-\sqrt{x+1}}{x} &= \frac{1}{x}\left(\frac{x}{\sqrt{2x+1}+\sqrt{x+1}}\right), \text{ where } x \neq 0 \\
&= \frac{1}{\sqrt{2x+1}+\sqrt{x+1}}, \text{ where } x \neq 0
\end{aligned}
$$