1. Length = 7 Dimension = 3

2. $\mathbf{G}^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$

3. YES
Since C has a generating matrix in standard form (see $\mathbf{G}^* = (\mathbf{I}_3\ \mathbf{X})$ above), it is a systematic code.

4. **(a)** 101 encoded using $\mathbf{G}^*$ gives 1010101.

(b) The first three digits of the codeword give the message word.

(c) 101 encoded using $\mathbf{G}$ gives 1100011.

(d) Let the message word be abc. Then

$$(a\ \ b\ \ c)\mathbf{G} = (a+b\quad a\quad a+b+c\quad 0\quad b\quad c\quad b+c).$$

Given a codeword of length seven, the corresponding message word will be the second digit, followed by the fifth digit, followed by the sixth digit.

5. $\mathbf{H} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

6. **(a)** Distance of C is 3.
We use Theorem 2.42 to make the following statements. There is no single row of zeros in $\mathbf{H}$, so $\delta \neq 1$. There is no pair of equal rows in $\mathbf{H}$, so $\delta \neq 2$. Rows 1, 5 and 6 sum to zero, so $\delta = 3$ (there are many other pairs of three rows that sum to zero).

(b) C is 2-error detecting and 1-error correcting.

7. Since C is 1-error correcting, for every codeword $v \in C$, the set $L(v)$ contains **at least** the codeword v and all seven words that are at distance one from v.

$$\Phi_p(v,v) = p^7 \text{ and if } d(u,v) = 1 \text{ then } \Phi_p(v,u) = p^6(1-p)$$

Hence

$$\Theta_p(C) = \sum_{w \in L(v)} \Phi_p(v,w) \geq p^7 + 7p^6(1-p).$$

8. An SDA for C is given below.

coset leader(s)	syndrome
0000000	0000
1000000	0110
0100000	0101
0010000	0011
0001000	1000
0000100	0100
0000010	0010
0000001	0001
0001100	1100
0001010	1010
0001001	1001
1001000	1110
0101000	1101
0011000	1011
1000001, 0100010, 0010100	0111
1001001, 0101010, 0011100	1111

How the SDA was constructed:
We know that C is 1-error correcting, so every word of weight zero or one must be a unique coset leader. We write these eight coset leaders in the left-hand column and their corresponding syndromes (0000 and the eight rows of $\mathbf{H}$) in the right-hand column.
There are $2^4 = 16$ cosets of C, so we are missing 8 syndromes. Write the missing syndromes (3 words of weight 2, 4 words of weight 3, and 1 word of weight 4) in the right-hand column.
Since coset leaders are words of least weight, we now look for coset leaders of length seven and weight two. Thus we are looking for pairs of rows of $\mathbf{H}$ that sum to the missing syndromes. Six of the eight missing syndromes have a 1 in the first position, so these syndroms are the unique sum of row 4 plus one other row. Thus these syndromes have unique coset leaders.
There are three pairs of rows of $\mathbf{H}$ that sum to 0111, so that syndrome has three coset leaders.
There are no pairs of rows that sum to the remaining syndrome 1111, so we look for sets of three rows that sum to 1111. Each set must include row 4 as that is the only way to get a 1 in the first position of the syndrome. There are three sets of three rows that sum to 1111, so this syndrome has three coset leaders.
We have considered possible coset leaders in order of increasing weight and every coset has at least one coset leader so we have a complete SDA.

9. The set $L(\mathbf{0})$ consists of the unique coset leaders. There are the words of length 7 and weight 0 or 1 as well as the words

$$\{0001100,\ 0001010,\ 0001001,\ 1001000,\ 0101000,\ 0011000\}.$$

Thus $\Theta_p(C) = p^7 + 7p^6(1-p) + 6p^5(1-p)^2$.

10. We use the following table to assist in decoding.

received word	syndrome	number of errors	most likely message word(s)	corresponding English letter(s)
1000110	0000	0	100	A
0111010	1100	2 (4th and 5th bits)	011	L
1111110	1110	2 (1st and 4th bits)	011	L
0100000	0100	1 (2nd bit)	000	space
1110111	0111	2 (1st and 7th bits) (2nd and 6th bits) (3rd and 5th bits)	 011 101 110	 L D O
1100011	0000	0	110	O
1111000	1000	1 (4th bit)	111	N
1100101	0110	1 (1st bit)	010	E

The most likely intended English message is: ALL DONE