Review Problems MATH2400

(1) Calculate $\liminf_{n\to\infty} a_n$, $\lim_{n\to\infty} a_n$, $\limsup_{n\to\infty} a_n$ for sequences $\{a_n\}$, where: (note: $\lim_{n\to\infty} a_n$ may not exist)

a) $a_n = \cos\left(\frac{n\pi}{8}\right)$ b) $a_n = \left(1 + \frac{(-1)^n}{n}\right)^n$ c) $a_n = \frac{4(-1)^{3n+1} n^2 + 1}{n^2+7}$

(2) Classify the following series as absolutely convergent, divergent, or conditionally convergent:

a) $a_n = 2^n - n$ b) $a_n = \frac{1}{2^n + n}$ c) $\frac{n^n}{n!}$ d) $\frac{(-1)^{n+1}}{n^4}$

e) $a_n = \frac{3^n+1}{4^n+5}$ f) $\frac{(-1)^n n^2}{2^n}$

(3) Given an ε-δ proof for the continuity of the following functions at $x=2$

a) $f(x) = x^2 + 7$ b) $f(x) = \frac{1}{x+4}$

(4) a) Let I be an interval in $\mathbb{R}$. Let J be a subinterval (ie J is an interval, $J \subset I$). Let f be uniformly continuous on I. Show that f is uniformly continuous on J. Note: you need the following extension of the definition of continuity ("one-sided continuity"):

$f: [a,b) \to \mathbb{R}$ is continuous (from the right) at a if: given $\varepsilon > 0$ $\exists$ $\delta > 0$:

$a \le x < a + \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

Analogous for continuity (from the left) at b for $f: (a,b] \to \mathbb{R}$. These definitions can be unified: $f: I \to \mathbb{R}$ is continuous on I if $\forall x_0 \in I$ given $\varepsilon > 0$ $\exists \delta > 0$:

$|x - x_0| < \delta$, $x \in I \Rightarrow |f(x) - f(x_0)| < \varepsilon$

b) Show that $f(x) = x^2$ is uniformly continuous on any interval of the form (a,b), $a < b \in \mathbb{R}$.

(5) Is $f(x) = 1/x$ uniformly cts on $(0,1]$?