INTERNAL STUDENTS ONLY

THE UNIVERSITY OF QUEENSLAND

CANDIDATES MUST NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

Mid Semester Examination, 11 April, 2013

MATH2400

Mathematicial Analysis

(Unit Courses)

Sol'n's

Time: 45 Minutes for working

No perusal time before examination begins

CREDIT WILL BE GIVEN ONLY FOR WORK WRITTEN ON THIS EXAMINATION SCRIPT.
FULL WORKING MUST BE SHOWN.
Use the back pages if the space provided is insufficient, and/or for rough working.
Answer **all** questions. Each question is worth 25 marks.
Check that this examination paper has 10 printed pages.
Students will be permitted one page (single sided) of hand-written notes. These notes must be written and signed by the student. No printed matter, mechanical copies or notes written by others will be permitted.
Hand-held calculators are allowed, but only the Casio FX82 series, or University approved (i.e. labelled).

FAMILY NAME (PRINT): ______________________

GIVEN NAMES (PRINT): ______________________

STUDENT NUMBER: | | | | | | | – | | | |

SIGNATURE: ______________________

EXAMINER'S USE ONLY			
QUESTION	MARK	QUESTION	MARK
1		3	
2		4	
TOTAL MARKS			

1. For the sequence $\{a_n\}$, with a_n as given below, calculate $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$.

   a) $a_n = \dfrac{(-1)^n n^4 - 7n}{4n^4 + n^3}$;

   b) $a_n = \dfrac{(-1)^n n^4 - 7n}{4n^6 + n^3}$;

   c) $a_n = \begin{cases} 2^{-n} & n \text{ even,} \\ (-1)^{(n-1)/2}\left(1 - \frac{1}{n}\right) & n \text{ odd.} \end{cases}$

a) $a_n = \dfrac{(-1)^n - 7/n^3}{4 + 1/n} \to \begin{cases} -1/4 & n \text{ odd} \\ 1/4 & n \text{ even} \end{cases}$

$\Rightarrow \limsup\limits_{n\to\infty} a_n = 1/4, \quad \liminf\limits_{n\to\infty} a_n = -1/4$

b) $a_n = \dfrac{(-1)^n - 7}{4n^2 + 1/n} \to 0$ as $n \to \infty$

$\Rightarrow \limsup\limits_{n\to\infty} a_n = \liminf\limits_{n\to\infty} a_n = 0.$

c) ~~For~~ $a_{2n} = 2^{-2n} \to 0$

$a_{2n+1} = (-1)^n\left(1 - \frac{1}{n}\right) \to \begin{cases} -1 & n \text{ odd} \\ 1 & n \text{ even} \end{cases}$

these are the only possible limits of convergent subsequences (any subsequence must contain ∞ly many terms o.t.f. a_{2n}, $a_{2(2n)+1}$ or $a_{2(2n+1)+1}$), so

$\limsup\limits_{n\to\infty} a_n = 1, \quad \liminf\limits_{n\to\infty} a_n = -1.$

2. **a)** Calculate $\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$

b) Prove: $\sum_{n=1}^{\infty}\frac{1}{4n^2-1}=\frac{1}{2}$. (If you can't prove this, at least try to show that the series converges).

a) $0 < \sqrt{n+1}-\sqrt{n} = \frac{(\sqrt{n+1}-\sqrt{n}).(\sqrt{n+1}+\sqrt{n})}{(\sqrt{n+1}+\sqrt{n})}$

$$= \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2\sqrt{n}}$$

$\to 0$ as $n \to \infty$.

b) The series $= \sum a_n$, where

$$a_n = \frac{1}{4n^2-1} = \frac{1/2}{2n-1} - \frac{1/2}{2n+1}.$$

Hence $S_k = \sum_{n=1}^{k} a_k$ (kth partial sum)

$$= \frac{1/2}{2(1)-1} - \cancel{\frac{1/2}{2(1)+1}} + \cancel{\frac{1/2}{2(2)-1}} - \cancel{\frac{1/2}{2(1+1)}} + \dots + \cancel{\frac{1/2}{2k-1}} - \frac{1/2}{2k+1}$$

(telescoping sum)

$$= \frac{1}{2} - \frac{1/2}{2k+1} \to \frac{1}{2} \text{ as } k \to \infty.$$

3. Give an ε–δ proof that $\lim\limits_{x \to 1}(x^2 - 4x) = -3$.

WTS: given $\varepsilon > 0 \ \exists \delta > 0$:

$|x - 1\cancel{3}| < \delta \Rightarrow |x^2 - 4x - -3| < \varepsilon \quad \oplus$

i.e. $|x^2 - 4x + 3| < \varepsilon$

i.e. $\underbrace{|x-1|}_{\text{small}} \ \underbrace{|x-3|}_{\text{bded}} < \varepsilon \quad \circledast$

Choose $\delta < 1$: $\Rightarrow |x-1| < 1$

i.e. $0 < x < 2$

so $-3 < x-3 < 1$ ie $|x-3| < 3$.

Hence choose $\delta < \min\{1, \varepsilon/3\}$ to see $\circledast$ is fulfilled, showing $\oplus$.

4. Classify the following series as absolutely convergent, conditionally convergent or divergent.

   (i) $\frac{1}{3}+\frac{1}{7}+\left(\frac{1}{3}\right)^2+\left(\frac{1}{7}\right)^2+\left(\frac{1}{3}\right)^3+\left(\frac{1}{7}\right)^3+\cdots$

   (ii) $\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n\left(2+\frac{2}{n}\right)^n$

   (iii) $\sum_{n=1}^{\infty}\frac{(-1)^n}{2n^2-n}$.

(i) This is Σa_n, with $a_n = \begin{cases} \left(\frac{1}{3}\right)^{n/2} = \frac{1}{(\sqrt{3})^n} & n \text{ even} \\ \left(\frac{1}{7}\right)^{(n-1)/2} = \left(\frac{1}{\sqrt{7}}\right)^n \frac{1}{\sqrt{7}} & n \text{ odd} \end{cases}$

so $0<|a_n|\leq\left(\frac{1}{\sqrt{3}}\right)^n$ $\forall n$, so $\limsup_{n\to\infty}|a_n|^{1/n}\leq\frac{1}{\sqrt{3}}<1$

$\Rightarrow$ Σa_n abs. conv. by root test.

(ii) $a_n=(-1)^n(1+\frac{1}{n})^n \Rightarrow |a_n|=(1+\frac{1}{n})^n\to e$ as $n\to\infty$.
Since $\lim_{n\to\infty} a_n \neq 0$, Σa_n must diverge (nth term test).

(iii) $|a_n|=\frac{1}{2n^2-n}<\frac{1}{2n^2-n^2}$ $\forall n\geq 1$
$=\frac{1}{n^2}$ & $\Sigma\frac{1}{n^2}$ converges (p test). Since $0<|a_n|<\frac{1}{n^2}$, $\Sigma|a_n|$ converges by comparison / squeeze, hence ~~so does~~ Σa_n is abs. conv.