INTERNAL STUDENTS ONLY

THE UNIVERSITY OF QUEENSLAND

CANDIDATES MUST NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

Mid Semester Examination, 4 April, 2012

MATH2400

Mathematicial Analysis

(Unit Courses)

Solutions

Time: **45 Minutes** for working

No perusal time before examination begins

CREDIT WILL BE GIVEN ONLY FOR WORK WRITTEN ON THIS EXAMINATION SCRIPT.
FULL WORKING MUST BE SHOWN.
Use the back pages if the space provided is insufficient, and/or for rough working.
Answer **all** questions. Each question is worth 25 marks.
Check that this examination paper has 9 printed pages.
NO programmable, graphing or ASCII calculators allowed.

FAMILY NAME (PRINT): ______________________

GIVEN NAMES (PRINT): ______________________

STUDENT NUMBER:

SIGNATURE: ______________________

EXAMINER'S USE ONLY			
QUESTION	MARK	QUESTION	MARK
1		3	
2		4	
TOTAL MARKS			

1. For the sequence $\{a_n\}$, with a_n as given below, calculate $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$.

a) $a_n = \dfrac{(-1)^n n^3 + 2n}{4n^2+9}$;

b) $a_n = \dfrac{(-1)^{2n+1} n^5}{4n^5+9}$;

c) $a_n = \left(1 - \dfrac{(-1)^n}{n}\right)^n$.

a) $$a_n = \frac{((-1)^n n^3 + 2n)/n^2}{(4n^2+9)/n^2} = \frac{(-1)^n n + 2/n}{4 + 9/n^2}$$

$$= \begin{cases} \dfrac{n + 2/n}{4 + 9/n^2} \to \infty & n \text{ even} \\ \dfrac{-n + 2/n}{4 + 9/n^2} \to -\infty & n \text{ odd} \end{cases}$$

So $\limsup = \infty$, $\liminf = -\infty$.

b) $(-1)^{2n+1} = 1$, So $a_n = \dfrac{-n^5}{4n^5+9} = \dfrac{-1}{4 + 9/n^5}$

$\to -1/4$ as $n \to \infty$.

c) n odd: $a_n = (1 + \frac{1}{n})^n \to e$ as $n \to \infty$

n even: $a_n = (1 - \frac{1}{n})^n = (\frac{n-1}{n})^n = (\frac{n}{n-1})^{-n}$

$= (1 + \frac{1}{n-1})^{-[(n-1)+1]}$ $= \left[(1 + \frac{1}{n-1})^{(n-1)+1}\right]^{-1}$

$\to e^{-1}$

since any subseq. must contain only many odd or even-indexed terms, e & e^{-1} are the only cluster pts:

So $\limsup = e$, $\liminf = e^{-1}$.

2. Suppose that the sequences $\{a_n\}$ and $\{b_n\}$ are convergent, with $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Suppose further $a_n < b_n$ for all n. What conclusions can you draw about a and b? Justify your claim carefully (i.e., use an $\varepsilon - N$ argument).

Claim: $a \leq b$.

Pf: $a_n \to a$, $b_n \to b \Rightarrow$ given $\varepsilon > 0$

$\exists\, N > 0 : \quad n \geq N \Rightarrow$

$$|a_n - a| < \varepsilon/2, \text{ i.e., } a - \varepsilon/2 \overset{*}{<} a_n < a + \varepsilon/2 \quad (1)$$

$$\& \ |b_n - b| < \varepsilon/2, \text{ i.e., } b - \varepsilon/2 < b_n \overset{*}{<} b + \varepsilon/2 \quad (2)$$

Then $\quad a - b < a_n + \varepsilon/2 - (b_n - \varepsilon/2)$ by $(1)^*, (2)^*$

$$= a_n - b_n + \varepsilon$$

$$< \varepsilon$$

This holds $\forall \varepsilon > 0$, so

$$a - b \leq 0, \text{ i.e., } a \leq b.$$

3. Give an ε–δ proof that $\lim_{x\to 1}(x^2 - 2x) = -1$.

WTS: Given $\varepsilon > 0$ $\exists \delta > 0$:
$|x-1| < \delta \Rightarrow |f(x) - -1| < \varepsilon$

i.e. $|x^2 - 2x + 1| < \varepsilon$

i.e. $|x-1|\,|x-1| < \varepsilon$ (*)

For $\delta < 1$, there holds

$|x-1| < 1$.

So for $|x-1| < \min\{1, \varepsilon\}$, (*)
holds: so choose $\delta < \min\{1, \varepsilon\}$.

4. Classify the following series as absolutely convergent, conditionally convergent or divergent.

(i) $\frac{1}{2} + \frac{1}{5} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{5}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{5}\right)^3 + \cdots$

(ii) $\sum_{n=1}^{\infty} (-1)^n \left(1 + \frac{1}{n}\right)^n$

(iii) $\sum_{n=1}^{\infty} a_n$, where $a_1 = 1$, and $a_{n+1} = a_n \left(\frac{1}{4} + \frac{(-1)^n}{2}\right)$.

(i) note $a_n > 0$, $a_n = \left(\frac{1}{2}\right)^{\frac{n+1}{2}}$ n odd, $\left(\frac{1}{5}\right)^{n/2}$ n even.

root test:

$$\sqrt[n]{a_n} = \begin{cases} \left(\frac{1}{2}\right)^{1/2 + 1/2n} & n \text{ odd} \\ \left(\frac{1}{5}\right)^{1/2} & n \text{ even} \end{cases}$$

$$\to \begin{cases} 1/\sqrt{2} & n \text{ odd} \\ 1/\sqrt{5} & n \text{ even} \end{cases}$$

$\Rightarrow \limsup_{n\to\infty} \sqrt[n]{a_n} = 1/\sqrt{2} < 1 \Rightarrow a_n$ is abs conv.

(ii) $|a_n| = \left(1 + \frac{1}{n}\right)^n \to e \neq 0$ as $n \to \infty$, so series diverges (nth term test)

(iii) $$\left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} 1/4 & n \text{ odd} \\ 3/4 & n \text{ even} \end{cases}$$

so $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, so series is abs. conv. by ratio test.