INTERNAL STUDENTS ONLY

THE UNIVERSITY OF QUEENSLAND

CANDIDATES MUST NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

Mid Semester Examination, 14 April, 2010

MATH2400

Mathematicial Analysis

(Unit Courses)

Solutions.

Time: 45 Minutes for working

No perusal time before examination begins.

CREDIT WILL BE GIVEN ONLY FOR WORK WRITTEN ON THIS EXAMINATION SCRIPT.

FULL WORKING MUST BE SHOWN.

Use the back pages if the space provided is insufficient, and/or for rough working.

Answer all questions. Each question is worth 25 marks.

Check that this examination paper has 9 printed pages.

NO programmable, graphing or ASCII calculators allowed.

FAMILY NAME (PRINT): ______________________

GIVEN NAMES (PRINT): ______________________

STUDENT NUMBER: | | | | | | | – | | |

SIGNATURE: ______________________

EXAMINER'S USE ONLY			
QUESTION	MARK	QUESTION	MARK
1		3	
2		4	
TOTAL MARKS			

1. For the sequence $\{a_n\}$, with a_n as given below, calculate $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$.

   a) $a_n = \dfrac{(-1)^n n^2 + 2n}{4n^2+9}$;

   b) $a_n = \dfrac{(-1)^{2n+1} n^4}{4n^2+9}$;

   c) $a_n = \left(1 + \dfrac{(-1)^n}{n}\right)^n$;

a) $a_n = \dfrac{(-1)^n n^2/n^2 + 2n/n^2}{4n^2/n^2 + 9/n^2} = \dfrac{(-1)^n + 1/n}{4 + 9/n^2} \Rightarrow \begin{cases} -1/4 & n \text{ odd} \\ 1/4 & n \text{ even.} \end{cases}$

So $\limsup_{n\to\infty} a_n = 1/4$, $\liminf_{n\to\infty} a_n = -1/4$.

b) $a_n = \dfrac{(-1)^{2n+1} n^4/n^2}{4n^2/n^2 + 9/n^2} = \dfrac{(-1)n^2}{4 + 9/n^2} \to -\infty$ as $n \to \infty$

So $\lim_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = -\infty$.

c) For n even: $a_n = \left(1 + \frac{1}{n}\right)^n \to e$.

For n odd: $a_n = \left(1 - \frac{1}{n}\right)^n = \left(\frac{n-1}{n}\right)^n = \left[\left(\frac{n}{n-1}\right)^n\right]^{-1}$

$= \left[\left(1 + \frac{1}{n-1}\right)^{n-1} \cdot \left(1 + \frac{1}{n-1}\right)\right]^{-1}$

$\to (e \cdot 1)^{-1} = e^{-1}$ as $n \to \infty$.

So $\liminf_{n\to\infty} a_n = e^{-1}$, $\limsup_{n\to\infty} a_n = e$.

2. Give an ε–δ proof of the continuity of the function $f(x) = x^3 - x$ at $x = 1$.

Note $f(1) = 0$. So given $\varepsilon > 0$ want $\delta > 0$
s.t. $|x-1| < \delta \Rightarrow |x^3 - x - 0| < \varepsilon$

i.e. $|x(x^2-1)| < \varepsilon$

i.e. $|x||x+1||x-1| < \varepsilon$. (*)

Choose $\delta \leq 1 \Rightarrow 0 < x < 2$ so $0 < |x||x+1| < 6$. (**)

So for $\delta < \min\{1, \varepsilon/6\}$ we have via (**) &

(*) $|f(x) - f(1)| < 6 \cdot \varepsilon/6 = \varepsilon$.

3. Show that the function $f(x) = \dfrac{x^2+5x-7}{(x^2-7x+6)(x-2)}$ is **uniformly** continuous on the interval $(3,4)$. (Hint: you don't need to calculated with ε's and δ's.)

Note that f is defined & cts where the denominator is nonzero, as it is a rational function. So f is continuous on $\mathbb{R} \setminus \{1,2,6\} = (-\infty, 1) \cup (1,2) \cup (2,6) \cup (6,\infty)$.
In particular, it is continuous on $(2,6)$, and hence on $[3,4]$, which is a subinterval of $(2,6)$. But $[3,4]$ is closed & bdded, so by Thm. 2.5 from the lectures, f is uniformly continuous on $[3,4]$. As $(3,4)$ is a subinterval of $[3,4]$, f is uniformly cts on $(3,4)$ (by Q4 on the practice sheet).

4. Suppose that the series $\sum_{n=1}^{\infty} a_n$ converges.

a) Show that given any $\varepsilon > 0$ there exists $N > 0$ such that $\sum_{n=N}^{\infty} a_n < \varepsilon$.

b) For $a_n = \frac{1}{2^n}$ you know that the series converges (geometric series, $|r| < 1$). For $\varepsilon = \frac{1}{2^{11}}$, find a suitable N.

a) Suppose $\sum_{n=1}^{\infty} a_n = S$. Put $S_k = \sum_{n=1}^{k} a_n$ (so S_k is the kth partial sum). Then $S_k \to S$ as $k \to \infty$ (as a sequence). So given $\varepsilon > 0$ $\exists N_0 > 0$ s.t.

$$k > N_0 \Rightarrow |S_k - S| < \varepsilon$$

$$\text{i.e. } \left|\sum_{n=k+1}^{\infty} a_n\right| < \varepsilon$$

In particular for $k = N_0+1$ this is true, & hence for $N = k+1 = N_0+2$:

$$\left|\sum_{n=N}^{\infty} a_n\right| < \varepsilon, \text{ so certainly } \sum_{n=N}^{\infty} a_n < \varepsilon.$$

b) $\sum_{n=1}^{\infty} a_n = 1$, & $\sum_{n=N}^{\infty} a_n = 2^{-(N-1)} \sum_{n=1}^{\infty} a_n = 2^{-(N-1)}$

So for $N = 13$, $\sum_{n=N}^{\infty} a_n = 2^{-12} < 2^{-11} = \varepsilon$.