MATH 2400 Final 2007

1) (i) $\lim_{n\to\infty} x^{2n} = 0 \quad (0 \leq x < 1)$

$\lim_{n\to\infty} 1^n = 1$

$\Rightarrow f = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$

(ii) No. If it were, the limit f on a closed, bded interval would have to be continuous (uniform limit of continuous functions is continuous), which it isn't.

(iii) $[0,p) \subset [0,1]$ & $f_n \to f$ on $[0,1]$, so $f_n \to f$ on $[0,p)$

(iv) see last page.

2) $f \in C^1[0,1]$. By Rolle's theorem, if f has 2 distinct roots a & b in $[0,1]$, with $0 \leq a < b \leq 1$, then $\exists\ c \in (a,b)$ (so $c \in (0,1)$) with $f'(c) = 0$ $(= f(a) = f(b))$. But $f'(c) = 3x^2 - 3 = 3(x^2-1)$ is strictly negative on $(0,1)$, which is a contradiction.

3) (i) $\frac{-1}{n^2} \leq \frac{(\sin x)^n}{n^2} \leq \frac{1}{n^2}$

So $0 \leq \frac{|(\sin x)^n|}{n^2} \leq \frac{1}{n^2}$. Since $\sum \frac{1}{n^2}$ converges (p-series, $p > 1$), $\frac{|(\sin x)^n|}{n^2}$ converges, ~~absolutely~~ meaning that $\sum \frac{(\sin x)^n}{n^2}$ converges absolutely.

(ii) $f(x) = \frac{1}{x\log x}$ is mono. decreasing & positive on $[3,\infty)$, ~~with $\lim_{x\to\infty} \frac{1}{x\log x} = 0$. So by the integral test,~~

$\sum \frac{1}{n\log n}$ converges $\Leftrightarrow \int_3^\infty \frac{dx}{x\log x}$ converges. Put $y = \log x \Rightarrow dy = \frac{dx}{x}$

$\int_3^\infty \frac{dx}{x\log x} = \int_*^\infty \frac{dy}{y} = \infty$. So series diverges.

(only positive terms, so it can't be conditionally convergent)

(iii) Analogous to (ii), as $x \mapsto \frac{1}{x(\log x)^2}$ is monotone decreasing, to 0 (as $n \to \infty$) $\int_3^\infty \frac{dx}{x(\log x)^2} \underset{=}{y=\log x} \int_*^\infty \frac{dy}{y^2}$

which converges, so the series converges absolutely

(iv) $0 < \frac{1}{n^2 \log n} < \frac{1}{n^2}$ for $n \geq 3$

So $\sum \frac{1}{n^2 \log n}$ converges absolutely* since $\sum \frac{1}{n^2}$.

* by the comparison test.

4) (i) $f^{(n)}(x) = \sinh x$ n even $\Rightarrow f^{(n)}(0) = 0$ n even
$\cosh x$ n odd $\quad 1$ n odd

$\Rightarrow$ Maclaurin series is $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$x + \frac{x^3}{3!} = P_3 = P_4$, so the error can be written as ~~estimated~~

~~by~~ $\frac{f^{(5)}(\xi)}{5!} x^5$ for some $\xi \in (0,x)$ or $(x,0)$, i.e.

the error is given by $R = \frac{\cosh \xi}{5!} x^5$ for some such

ξ, so $|R| \leq \frac{\cosh .5}{2^5 \cdot 5!} = .00029$.

5) (i) For $f(x,y,z) = \frac{xy + xz}{x^2+y^2+z^2}$ & $(x,y,z) = (\alpha t, \beta t, 0)$ with $t \neq 0$

we have $f(\alpha t, \beta t, 0) = \frac{\alpha\beta}{\alpha^2+\beta^2} \Rightarrow \lim_{t \to 0} f(\alpha t, \beta t, 0)$

depends on α & β (e.g. $\alpha = \beta = 1$: limit $= \frac{1}{2}$; $\alpha = 1, \beta = 0$
limit $= 0$), so $\lim_{(x,y,z) \to (0,0,0)} f(x,y,z)$ doesn't exist.

5 b) Put $f(x,y) = \frac{x^4 - y^4}{x - y}$. Since $f(x,y)$ is undefined for $x = y$, f is not defined on any deleted nbhd of $(2,2)$, so the limit is not defined, according to our definition.

However, full marks were also given for noting that for $x \neq y$, $f(x,y) = \frac{\cancel{(x-y)}(x^3 + x^2y + xy^2 + y^3)}{\cancel{(x-y)}}$,

$$\Rightarrow \lim_{\substack{(x,y)\to(2,2) \\ (x \neq y)}} f(x,y) = \lim_{\substack{(x,y)\to 2,2 \\ (x \neq y)}} (8 + 8 + 8 + 8) = 32$$

6 (i) $J_f = \begin{pmatrix} 6x^5 & -6y^5 \\ y & x \end{pmatrix}$ $\det J_f = 6(x^6 + y^6)$

(ii) $f : \mathbb{R}^2 \to \mathbb{R}^2$ is C^1, with $\det J_f \neq 0$ for $(x,y) \neq (0,0)$

Inv. f^n Th^m $\Rightarrow$ f has a local inverse near any $(x,y) \neq (0,0)$.

(iii) $f(1,0) = (1,0) = f(-1,0) \Rightarrow f$ is not globally invertible.

1 (iv) In order to show unif. cvrgnce:

Given $\varepsilon > 0$ need $N \in \mathbb{N}$ s.t.

$n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon \quad \forall x \in [0, \rho)$,

i.e. $n \geq N \Rightarrow x^{2n} < \varepsilon \quad \forall x \in [0, \rho)$. (*)

So choose N s.t. $\rho^{2N} < \varepsilon$ (possible, since $x \mapsto x^{2N}$ is a decreasing f^n of N). For $n > N$, there holds: $0 \leq x^{2n} < \rho^{2n} < \rho^{2N} < \varepsilon \Rightarrow$ (*)