INTERNAL STUDENTS ONLY

THE UNIVERSITY OF QUEENSLAND

CANDIDATES MUST NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

First Semester Examination, June, 2003

MATH2400

MATHEMATICAL ANALYSIS

(Unit Courses)

Time: **TWO hours** for working

Ten minutes for perusal before examination begins

CREDIT WILL BE GIVEN ONLY FOR WORK WRITTEN ON THIS EXAMINATION SCRIPT.

Use the back pages if the space provided is insufficient, or for rough working.

Questions carry the number of marks shown. The total number of marks is 70.

Attempt all questions

Pocket calculators allowed.

FAMILY NAME (PRINT): _____

GIVEN NAMES (PRINT): _____

STUDENT NUMBER: | | | | | | | | |

SIGNATURE: _____

EXAMINER'S USE ONLY	
QUESTION	MARK
1	
2	
3	
4	
5	
6	
TOTAL	

TURN OVER

MATH2400 — MATHEMATICAL ANALYSIS
First Semester Examination, June, 2003 (continued)

1. (i) Find the limit of the sequence as n tends to infinity

$$a_n = \sqrt[n]{4^n + 5^n}.$$

(6 marks)

(ii) Let the sequence $\{a_n\}$ be defined recursively by $a_1 = 2$, $a_{n+1} = \frac{1}{2}(a_n + 4)$ for $n \geq 1$.

(a) Show by induction on n that $a_n < 4$ for each n and that $\{a_n\}$ is monotone increasing sequence.

(b) Show that this sequence converges and find its limit.

(9 marks)

Question 1 continued next page

MATH2400 — MATHEMATICAL ANALYSIS
First Semester Examination, June, 2003 (continued)

1. (Working space continued)

Question 2 see next page

MATH2400 — MATHEMATICAL ANALYSIS
First Semester Examination, June, 2003 (continued)

2. (i) Consider a sequence of functions $f_n : [a, b] \to \mathbb{R}$. Explain what is meant by each of the statements:

   (a) f_n converges to f pointwise on $[a, b]$.

   (b) f_n converges uniformly to f on $[a, b]$.

(6 marks)

(ii) Determine whether the following sequences converge. Are they also uniformly convergent? **(justify your answer)**:

(a) $\dfrac{\sin n\,x}{n}$ on $\mathbb{R}$;

(b) $\dfrac{1}{nx+1}$ on $[0, 1]$.

(9 marks)

Question 2 continued on next page

2. (Continued working page only)

Question 3 see next page

3. (i) State the intermediate value theorem for continuous functions.

(3 marks)

(ii) Let $f : [1, 2] \longrightarrow [0, 3]$ be a continuous function satisfying $f(1) = 0$ and $f(2) = 3$. Show that there exists a point $x_0 \in [1, 2]$ such that $f(x_0) = x_0$.

(7 marks)

Question 3 continued next page

3. (Continued working page only)

Question 4 see next page

MATH2400 — MATHEMATICAL ANALYSIS
First Semester Examination, June, 2003 (continued)

4. (i) Do the following limits exist? **(Explain your answers.)**

   (a) $\lim_{(x,y,z)\to(0,0,0)} \dfrac{x^2y^2 + y^2z^2}{x^2 + y^2 + z^2}$,

   (b) $\lim_{(x,y)\to(2,2)} \dfrac{x^4 - y^4}{x - y}$.

(7 marks)

(ii) Consider a function $f : \mathbb{R}^n \to \mathbb{R}^m$. Explain what is meant by each of the statements:

(a) f is continuous at $x_0 \in \mathbb{R}^n$;

(b) f is differentiable at $x_0 \in \mathbb{R}^n$.

(3 marks)

Question 4 continued next page

4. (Continued working page only)

5. (a) State the inverse function theorem.

(3 marks)

(b) Let $f(x, y) = (x^2y + yx, 2xy + xy^2)$.
Show that f, in an open set containing $(1, 1)$, has an inverse f^{-1} defined in an open set containing $(2, 3)$. Find the Jacobian matrix of f^{-1} at $(2, 3)$.

(7 marks)

Question 5 continued next page

5. (Continued working page only)

6. (a) Calculate the Jacobian determinant of the transformation from spherical to rectangular coordinates given by

$$(x,\, y,\, z) = T(r,\, \varphi,\, \theta) = (r\sin\varphi\cos\theta,\, r\sin\varphi\sin\theta,\, r\cos\varphi).$$

(4 marks)

(b) Evaluate $\iiint_V z\,dx\,dy\,dz$, where V is the octant of the unit ball defined by

$$x^2 + y^2 + z^2 \leq 1, \quad x \geq 0, \quad y \geq 0, \quad z \geq 0.$$

(7 marks)

Question 6 continued next page

6. (Continued working page only)