MATH2400 Assignment 5 Solutions

20th May 2014

Question 1

Let $G(x)$ be the antiderivative of $g(x) = e^{x^2}$, i.e. $G'(x) = g(x) = e^{x^2}$. Then by the FTC we have

$$\int_0^{x^4} e^{t^2}\,dt = G(x^4) - G(0).$$

Taking the derivative of the RHS with respect to x gives

$$\begin{aligned} \frac{d}{dx}\big(G(x^4) - G(0)\big) &= G'(x^4)\cdot 4x^3 - G'(0)\cdot 0 \\ &= e^{x^8}\cdot 4x^3, \end{aligned}$$

hence

$$\frac{d}{dx}\int_0^{x^4} e^{t^2}\,dt = 4x^3 e^{x^8}.$$

Question 2

Recalling that the Taylor expansion of e^y around 0 is

$$e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!},$$

we make the substitutions $y = x^2$ and $y = 2x^3$ to get

$$e^{x^2} + e^{2x^3} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} + \sum_{n=0}^{\infty} \frac{2^n x^{3n}}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n} + 2^n x^{3n}}{n!}.$$

We now look at convergence. It will be easiest to consider each series individually. Using the ratio test we have

$$\lim_{n\to\infty} \frac{|x^{2(n+1)}|}{|(n+1)!|}\cdot\frac{|n!|}{|x^{2n}|} = \lim_{n\to\infty}\frac{x^2}{n} = 0$$

and

$$\lim_{n\to\infty} \frac{|2^{n+1}x^{3(n+1)}|}{|(n+1)!|}\cdot\frac{|n!|}{|2^n x^{3n}|} = \lim_{n\to\infty}\frac{|2x^3|}{n} = 0$$

for all fixed $x \in \mathbb{R}$. Hence both series converge for all $x \in \mathbb{R}$ and the Taylor series converges to $f(x)$ on all of $\mathbb{R}$.

Question 3

We first note that

$$0 \leq \frac{n^2 + \cos n}{e^{n^3}} \leq \frac{3n^2}{e^{n^3}} = 3n^2 e^{-n^3}.$$

We aim to use the integral test, so calculating the appropriate integral gives

$$\begin{aligned}\int_1^\infty 3x^2 e^{-x^3} dx &= \lim_{a \to \infty} \left(-e^{-a^3} + e^{-1} \right) \\ &= \frac{1}{e}.\end{aligned}$$

Hence by the integral test, the series

$$\sum_{n=1}^{\infty} \frac{n^2 + \cos n}{e^{n^3}}$$

converges (absolutely).

Question 4

Expanding $\sinh(x)$ around 0 we have

$$\begin{aligned}T_n(x) &= \frac{\sinh(0)}{0!} x^0 + \frac{\cosh(0)}{1!} x^1 + \frac{\sinh(0)}{2!} x^2 + \cdots + \frac{\sinh^{(n)}(0)}{n!} x^n \\ &= 0 + x^1 + 0 + \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 + \cdots + \frac{\sinh^{(n)}(0)}{n!} x^n.\end{aligned}$$

Note that the even terms are 0, hence for n odd we have

$$T_n(x) = T_{n+1}(x),$$

i.e. if we calculate up to n odd, we get the next term for free. So assume we have n even. Then our error term is

$$R_n(x) = \frac{\sinh(c)}{(n+1)!} x^{n+1}$$

which at $x = 1$ becomes

$$R_n(1) = \frac{\sinh(c)}{(n+1)!}$$

for some $c \in (0, 1)$. For $c \in (0, 1)$ we can (crudely) bound $\sinh(c)$ by

$$0 \leq \sinh(c) \leq 2,$$

hence

$$0 \leq R_n(1) \leq \frac{2}{(n+1)!}.$$

To ensure that our estimate is within 6 decimal places, we must choose an n large enough such that

$$\frac{2}{(n+1)!} < 10^{-7}.$$

Choosing $n = 10$ will suffice. Hence

$$\sinh(1) \approx 0 + \frac{1}{1!} + 0 + \frac{1}{3!} + 0 + \frac{1}{5!} + 0 + \frac{1}{7!} + 0 + \frac{1}{9!} + 0 = 1.17520116843.$$

Question 5

Part (a)

The integral test tells us that the series $\sum_{n=a}^{\infty} f(n)$ converges if and only if the improper integral $\int_a^\infty f(x)\,dx$ converges. Hence we wish to determine if the integral

$$\int_3^\infty \frac{1}{x\log(\log(\log(x)))}\,dx$$

converges. We first note that the integrand is well defined on $(3, \infty)$. We make the substitution $u = \log(\log(x))$. Then $\frac{du}{dx} = \frac{1}{\log(x)} \cdot \frac{1}{x}$, hence

$$\begin{aligned}\int_3^\infty \frac{1}{x\log(\log(\log(x)))}\,dx &= \int_{\log(\log(3))}^\infty \frac{1}{u}\,du \\ &= \lim_{b\to\infty} \left[\log(b) - \log(\log(\log(3)))\right].\end{aligned}$$

But since $\log(b) \to \infty$ as $b \to \infty$, the above limit does not exist and the series diverges.

Part (b)

We wish to compare our series to the harmonic series. Recalling that $\log(n) < n$ we observe that

$$\frac{1}{n + \log(n)} \geq \frac{1}{n+n} = \frac{1}{2n}.$$

But we know that the harmonic series diverges (by ratio test, for example), and that

$$\sum_{n=3}^\infty \frac{1}{n+\log(n)} \geq \sum_{n=3}^\infty \frac{1}{2n} = \frac{1}{2}\sum_{n=3}^\infty \frac{1}{n}.$$

Hence the series diverges by the comparison test.

Question 6

Suppose that the limit exists and is equal to $L \in \mathbb{R}$. Then for any parametrised curve $\gamma : \mathbb{R} \to \mathbb{R}^3$ where $\gamma(t) = (x(t), y(t), z(t))$ and $\gamma(0) = (0, 0, 0)$, we must have

$$\lim_{t \to 0} \frac{x^2yz}{x^8 + y^4 + z^3} = L.$$

As per the hint, calculating the limit as we approach along the curve (t, t^2, t^4) we get

$$\lim_{t \to 0} \frac{x^2yz}{x^8 + y^4 + z^3} = \lim_{t \to 0} \frac{t^8}{3t^8} = \frac{1}{3} = L.$$

But calculating the limit as we approach along the curve (t, t, t) we find

$$\lim_{t \to 0} \frac{x^2yz}{x^8 + y^4 + z^3} = \lim_{t \to 0} \frac{t^4}{t^8 + t^4 + t^2} = 0 \neq L.$$

Contradiction. Hence our assumption that the limit exists was incorrect.