MATH2400 Assignment 4 Solutions

Question 1:

We first note that

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

converges by the p test. Now, let $x \in \mathbb{R}$ be arbitrary. Then:

$$0 < \frac{e^{-x^2}}{n^3 + x^2} \leq \frac{1}{n^3 + x^2}$$

$$\leq \frac{1}{n^3}.$$

So by the comparison test, the series converges for all $x \in \mathbb{R}$. The series converges uniformly by the Weierstrass M-test. That is, given a sequence of functions f_n, if there is a sequence of positive numbers M_n satisfying, for all $x \in \mathbb{R}$,

$$1) \quad |f_n(x)| \leq M_n$$
$$2) \quad \sum_{n=1}^{\infty} M_n < \infty,$$

then the series

$$\sum_{n=1}^{\infty} f_n(x)$$

converges uniformly. Here, the sequence $(M_n) = \frac{1}{n^3}$, and as shown above, $|f_n(x)| \leq M_n$ for all x.

If you wanted to show the result directly: Define

$$f_k(x) = \sum_{n=1}^{k} \frac{e^{-x^2}}{n^3 + x^2},$$

with limit function

$$\lim_{k \to \infty} f_k(x) =: f(x) = \sum_{n=1}^{\infty} \frac{e^{-x^2}}{n^3 + x^2}.$$

Note we know $f(x)$ exists since we have just proven pointwise convergence. In order to show uniform convergence, we are required to show the following: Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $x \in \mathbb{R}$

$$|f_k(x) - f(x)| < \varepsilon,$$

whenever $k > N$.

Let $\varepsilon > 0$ be given. Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, by the definition of convergence of an infinite series, there exists some $N_0 \in \mathbb{N}$ such that

$$\begin{aligned} \varepsilon &> \left| \sum_{n=1}^{\infty} \frac{1}{n^3} - \sum_{n=1}^{k} \frac{1}{n^3} \right| \\ &= \sum_{n=k+1}^{\infty} \frac{1}{n^3}, \end{aligned} \tag{1}$$

for all $k > N_0$. Set $N > N_0$ and let $k > N$. Then, for *any* $x \in \mathbb{R}$, we have

$$\begin{aligned} |f_k(x) - f(x)| &= \left| \sum_{n=k+1}^{\infty} \frac{e^{-x^2}}{n^3 + x^2} \right| \\ &\leq \sum_{n=k+1}^{\infty} \frac{1}{n^3} \\ &< \varepsilon, \end{aligned}$$

by (1) since $k > N > N_0$. We note that the choice of N was independent of the choice of $x \in \mathbb{R}$. That is, the convergence is uniform. (One may see this argument is really why the Weierstrass M test works).

Question Two:

We first note that each f_n is continuous since it is the composition of continuous functions. Furthermore, f_n converges pointwise to a limit function f since

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^{2n}}{1+x^{2n}}$$

$$= \lim_{n\to\infty} \left[1 - \frac{1}{1+x^{2n}}\right]$$

$$= \begin{cases} 0, & 0 < x < 1, \\ \frac{1}{2}, & x = 1, \\ 1, & x > 1, \end{cases}$$

$$=: f(x).$$

We recall the definition of uniform convergence. Let $X \subseteq \mathbb{R}$. A sequence $(f_n : X \to \mathbb{R})_{n=1}^{\infty}$ of functions converges uniformly to a function $f : X \to \mathbb{R}$ if given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that *for any* $x \in X$,

$$|f_n(x) - f(x)| < \varepsilon,$$

whenever $n > N$.

We note in this instance, $0 \leq f_n(x), f(x) \leq 1$. Thus, to show uniform convergence, we need only consider $\varepsilon \in (0,1)$, since the case of $\varepsilon \geq 1$ will hold automatically.

Uniform convergence on $[0, a]$:

Let $a \in (0,1)$ be arbitrary. Set $N = \frac{\log \varepsilon}{2 \log a} = \frac{1}{2} \log_a \varepsilon$. Note that $N > 0$ since both $0 < a, \varepsilon < 1$. Let $n > N$. Then for *any* $x \in [0, a]$,

$$|f_n(x) - f(x)| = \left| \frac{x^{2n}}{1+x^{2n}} - 0 \right|$$

$$= \frac{x^{2n}}{1+x^{2n}}$$

$$\leq x^{2n}$$

$$\leq a^{2n}, \quad \text{since } x \leq a,$$

$$< a^{\log_a \varepsilon}, \quad \text{since } a < 1 \text{ and } n > N,$$

$$= \varepsilon.$$

So $f_n \to 0$ uniformly on $[0, a]$.

Uniform convergence on $[b, \infty)$:

Let $b \in (1, \infty)$ be arbitrary. Set $N > -\frac{1}{2}\frac{\log \varepsilon}{\log b} = -\frac{1}{2}\log_b \varepsilon$. Note that $N > 0$ since $b > 1$ and $0 < \varepsilon < 1$. Set $n > N$. Then for any $x \in [b, \infty)$,

$$\begin{aligned}
|f(x) - f_n(x)| &= \left|1 - \frac{x^{2n}}{1 + x^{2n}}\right| \\
&= \frac{1}{1 + x^{2n}} \\
&< \left(\frac{1}{x}\right)^{2n} \\
&\leq \left(\frac{1}{b}\right)^{2n}, \quad \text{since } \frac{1}{b} \geq \frac{1}{x} \\
&< \left(\frac{1}{b}\right)^{-\log_b \varepsilon}, \quad \text{since } 1/b < 1 \text{ and } n > N, \\
&= \varepsilon.
\end{aligned}$$

So $f_n \to 1$ uniformly on $[b, \infty)$.

Does the sequence of functions converge uniformly on $[0, \infty)$?

No.

In class, you proved the following theorem:

If $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous functions on $X \subset \mathbb{R}$ and $f_n \to f$ uniformly on X, then f is continuous. By contraposition of this statement, this is equivalent to: If f is not continuous then either the sequence of functions $\{f_n\}_{n=1}^{\infty}$ is not continuous or f_n does not converge uniformly to f. Now, if $f_n \to f$ uniformly, this limit function must also be its pointwise limit function. In our case, the pointwise limit function is not continuous. Thus, either the sequence of functions is not continuous, or the sequence does not converge uniformly to f. Since each f_n is continuous for all $x \in [0, \infty)$, it must be that the sequence does not converge uniformly.

Here is a different, more direct, argument to show the sequence of functions doesn't converge uniformly:

Suppose $f_n \to f$ uniformly on $[0, \infty)$. Then it must follow $f_n \to f$ uniformly on $[0, 1)$. So, by definition of uniform convergence, given $\varepsilon > 0$, there exists some $K \in \mathbb{N}$ such that *for all* $x \in [0, 1)$,

$$
\begin{aligned}
& |f_K(x) - f(x)| < \varepsilon \\
\iff & \left| \frac{x^{2K}}{1 + x^{2K}} - 0 \right| < \varepsilon, \quad \text{since } f \equiv 0 \text{ on } [0, 1), \\
\iff & \frac{x^{2K}}{1 + x^{2K}} < \varepsilon. \qquad (2)
\end{aligned}
$$

Let $\varepsilon = \frac{1}{4}$. Since (2) must hold for all $x \in [0, 1)$, it must hold for some

$$\left(\frac{1}{3}\right)^{\frac{1}{2K}} < x < 1.$$

But then,

$$
\begin{aligned}
& x > \left(\frac{1}{3}\right)^{\frac{1}{2K}} \\
\iff & x^{2K} > \frac{1}{3} \\
\iff & 3x^{2K} > 1 \\
\iff & 4x^{2K} > 1 + x^{2K} \\
\iff & \frac{x^{2K}}{1 + x^{2K}} > \frac{1}{4} \\
& = \varepsilon.
\end{aligned}
$$

Which contradicts (2).

Question Three:

There does not exist such a function. To prove this, we assume for sake of contradiction that there exists a continuously differentiable function on $[1, 5]$ such that $f(1) < 0$, $f(5) > 3$ and $f'(x) \leq e^{-f(x)}$.

Since f is differentiable, it is continuous. Thus, by the Intermediate Value Theorem, there exists at least one $x \in (1, 5)$ such that $f(x) = 2$. We now define

$$D := \{x \in (1, 5) | f(x) = 2\}.$$

Since D is bounded, $x_0 := \sup D$ exists. We now claim $x_0 \in D$. Proof: As discussed in lectures, for any $\varepsilon > 0$, there exists $x \in D$ such that $x > x_0 - \varepsilon$. From this fact, one is able to construct a sequence $(x_n) \subset D$ such that $x_n \to x_0$. Then, since f is continuous,

$$2 = f(x_n) \to f(x_0),$$

as $n \to \infty$. Since the limit of a constant is the constant itself, it follows that $f(x_0) = 2$ and thus $x_0 \in D$. We now claim: $f(x) \geq 2$ for any $x \in (x_0, 5)$. Proof: Suppose that $f(x_1) < 2$ for some $x_1 \in (x_0, 5)$. Then, by IVT, there exists $x_2 \in (x_1, 5)$ such that $f(x_2) = 2$. But then $x_2 \in D$ and $x_2 > x_0$. Contradiction - since x_0 is the supremum of D. So $f(x) \geq 2$ for all $x \in (x_0, 5)$. Since f is differentiable on $[1, 5]$, it is differentiable on $[x_0, 5]$. Thus, it obeys the Mean Value Theorem on $[x_0, 5]$. That is, there exists some $c \in (x_0, 5)$ such that

$$\begin{aligned} f'(c) &= \frac{f(5) - f(x_0)}{5 - x_0} \\ &> \frac{3 - 2}{5 - x_0}, \quad \text{since } f(5) > 3 \text{ and } f(x_0) = 2, \\ &> \frac{1}{5}, \quad \text{since } 1 < x_0 < 5. \end{aligned} \tag{3}$$

However, by assumption, f also satisfies the property that $f'(x) \leq e^{-f(x)}$ for all $x \in [1, 5]$. So,

$$\begin{aligned} f'(c) &\leq e^{-f(c)} \\ &\leq e^{-2}, \quad \text{since } f(x) \geq 2, \, \forall x \in (x_0, 5), \\ &< \frac{1}{7} \\ &< \frac{1}{5}. \end{aligned}$$

This is a contradiction. Thus, no such function satisfying the desired properties exists.

Here is another method. Suppose such a function exists. That is, suppose there exists a smooth function f such that

$$f(1) < 0 \tag{4}$$
$$f(5) > 3 \tag{5}$$
$$f'(x) \leq e^{-f(x)}, \quad \forall\, x \in [1,5]. \tag{6}$$

Consider the function $g(x) := e^{f(x)}$. Then, g is differentiable since it is a composition of differentiable functions. Thus, g obeys the mean value theorem on $[1,5]$. So, there exists some $c \in (1,5)$ such that

$$\begin{aligned} g'(c) &= \frac{g(5) - g(1)}{5 - 1} \\ &= \frac{e^{f(5)} - e^{f(1)}}{4} \\ &> \frac{e^3 - 1}{4} \\ &> 1, \end{aligned} \tag{7}$$

by assumptions (4) and (5).

However, we may calculate $g'(c)$ by the chain rule. We see

$$\begin{aligned} g'(c) &= f'(c)e^{f(c)} \\ &\leq e^{-f(c)}e^{f(c)} \\ &= 1, \end{aligned} \tag{8}$$

by assumption (6). But (7) and (8) yield a contradiction. So, there does not exist such a function.

Question Four:

Define the function

$$g(x) := \begin{cases} x, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

Claim: $\lim_{x \to 0} g(x) = 0$.

Proof: To prove the claim, we are required to show, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x| < \delta \Rightarrow |g(x)| < \varepsilon.$$

We immediately see

$$|g(x)| \leq |x|,$$

since equality holds when x is rational, and $|g(x)| = 0 < |x|$ when x is irrational. Take $\delta = \varepsilon$, and let $|x| < \delta$. Then

$$|g(x)| \leq |x|$$

$$< \delta = \varepsilon.$$

Thus, $\lim_{x \to 0} g(x) = 0$.

In order to show differentiability of f at 0, we are required to show

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x}$$

exists. But then, the above limit is just

$$\lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \begin{cases} \frac{x^2}{x}, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

$$= \lim_{x \to 0} g(x)$$

$$= 0,$$

as shown above. Thus, f is differentiable at 0 with $f'(0) = 0$.