14. Rational canonical form

1. If the characteristic polynomial of $T \in \ell(V)$ does not split, there is no basis on which $T$ has a diagonal form nor a Jordan canonical basis.

2. But the characteristic polynomial factorises as $(-1)^n(\phi_1(t))^{n_1}(\phi_2(t))^{n_2} \ldots (\phi_j(t))^{n_j}$.

3. There exists a basis $\beta$ such that

$$[T]_\beta = \begin{pmatrix}
C_1 & 0 & \cdots & 0 \\
0 & C_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & C_r
\end{pmatrix}$$

where each “0” is a zero matrix, and each $C_i$ is a square matrix of the form

$$C_i = \begin{pmatrix}
0 & 0 & \cdots & 0 & -a_0 \\
1 & 0 & \cdots & 0 & -a_1 \\
0 & 1 & \cdots & 0 & -a_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & -a_{k-1}
\end{pmatrix}$$
Definitions

A polynomial $f(t)$ is called **monic** if its leading coefficient is 1.

If $f(t)$ has positive degree and cannot be expressed as a product of positive-degree polynomials with coefficients in the field, we call $f(t)$ **irreducible**.

A matrix of the form $C_i$ (previous slide) is called a **companion matrix** of the monic polynomial

$$a_0 + a_1 t + \ldots + a_{k-1} t^{k-1} + t^k.$$ 

Use the notation $C_x$ for the $T$-cyclic subspace generated by $x \in V$. If $\dim(C_x) = k$, then \( \{x, T(x), T^2(x), \ldots, T^{k-1}(x)\} \) is a basis for $C_x$ called the **$T$-cyclic basis generated by** $x$, denoted $\beta_x$. We have that $\exists x \in V$ s.t. $C_i = [T_{C_x}]_{\beta_x}$. 

2
A matrix representation of the form $[T]_\beta$ (from the first slide) is called a **rational canonical form** of $T$, and the basis $\beta$ is called a rational canonical basis.

Each $C_i$ in the rational canonical form is a companion matrix of a polynomial $(\phi(t))^m$ such that $\phi(t)$ is an irreducible monic divisor of the characteristic polynomial and $0 < m \in \mathbb{Z}$.

**Main result**

For every linear operator on a finite dimensional vector space, there exists a rational canonical basis and hence a rational canonical form.

**Main problem**

Find a rational canonical basis.
**Definition**

Let $T$ be a linear operator on a finite dimensional vector space $V$ with characteristic polynomial

$$(-1)^n(\phi_1(t))^{n_1}(\phi_2(t))^{n_2} \ldots (\phi_k(t))^{n_k}$$

where the $\phi_i(t)$ are distinct irreducible monic polynomials and $0 < n_i \in \mathbb{Z}$. For $1 \leq i \leq k$ we define

$$K_{\phi_i} = \{x \in V \mid (\phi_i(T))^p(x) = 0, \text{ some } 0 < p \in \mathbb{Z}\}.$$

**Theorem**

$\beta$ is a rational canonical basis if and only if $\beta$ is the disjoint union of $T$-cyclic bases $\beta_{v_i}$, where each $v_i$ lies in $K_{\phi}$ for some irreducible monic divisor $\phi(t)$ of the characteristic polynomial of $T$. 
Example

$T \in \ell(\mathbb{R}^8)$, $\beta = \{v_1, v_2, \ldots, v_8\}$ such that

$$[T]_\beta = \begin{bmatrix}
0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}$$

Let $\phi_1(t) = t^2 - t + 3$ and $\phi_2(t) = t^2 + 1$. The diagonal blocks are companion matrices of the polynomials $\phi_1(t)$, $(\phi_2(t))^2$ and $\phi_2(t)$. 
**Definition**

Let $T \in \ell(V)$ for f.d. $V$. A polynomial $p(t)$ is called a **minimal polynomial** of $T$ if $p(t)$ is a monic polynomial of least degree such that $p(T) = T_0$.

**Theorem**

Suppose that the minimal polynomial of $T \in \ell(V)$ (f.d. $V$) is

$$p(t) = (\phi_1(t))^{m_1}(\phi_2(t))^{m_2} \cdots (\phi_k(t))^{m_k}$$

where the $\phi_i(t)$ are distinct irreducible monic factors of $p(t)$ and $0 < m_i \in \mathbb{Z}$. Then

$$K_{\phi_i} = \ker((\phi_i(T))^{m_i}).$$

If $\gamma_i$ is a basis for $K_{\phi_i}$, then $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ is a basis for $V$. Moreover, if each $\gamma_i$ is a disjoint union of $T$-cyclic bases, then $\gamma$ is a rational canonical basis.
How to form a rational canonical basis associated with $T$?

1. Determine the characteristic polynomial of $T$.
2. Determine the minimal polynomial of $T$.
3. Find $T$-cyclic bases associated to each $K_{\phi_i}$.
4. Form a rational canonical basis $\beta$ of $V$ as a disjoint union of these $T$-cyclic bases.
Example

\[ T \in \ell(P_3(\mathbb{R})) \text{ s.t. } T(f(x)) = f(0)x - f'(1). \]

Let \( \beta = \{1, x, x^2, x^3\} \)

\[
A = [T]_\beta = \begin{pmatrix}
0 & -1 & -2 & -3 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\( \Rightarrow \) characteristic polynomial \((t^2 + 1)t^2\).

Note \( A^2 + I = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -2 & -3 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \)

so \((A^2 + I)A = 0.\)

The minimal polynomial is therefore \((t^2 + 1)t.\)
To find $K_{t^2+1}$:

Find all $v$ s.t. $(A^2 + I)v = 0$.

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -2 & -3 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c \\
d
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

$\Rightarrow$ $a, b$ free and $c, d = 0$

$\Rightarrow \begin{Bmatrix}
\begin{pmatrix}
1 \\
0 \\
0 \\
0
\end{pmatrix},
\begin{pmatrix}
0 \\
1 \\
0 \\
0
\end{pmatrix}
\end{Bmatrix}$ is a basis for $K_{t^2+1}$

which is a $T$-cyclic basis.
To find $K_t$:

Find all $v$ s.t. $Av = 0$.

\[
\begin{pmatrix}
0 & -1 & -2 & -3 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c \\
d \\
\end{pmatrix}
=
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}
\]

$\Rightarrow a = 0$ and $b = -2c - 3d$

$\Rightarrow \left\{ \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $K_t$

which is the union of two disjoint bases with one vector each (since both are eigenvectors).
A rational canonical basis for $T$ (in coordinate vector form) is

$$\Rightarrow \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

which in terms of vectors in $P_3(\mathbb{R})$ is

$$\{1, x, x^2 - 2x, x^3 - 3x\}.$$ 

The corresponding rational canonical form is

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
Example

\[ T \in \ell(M_2(\mathbb{R})) \text{ s.t. } T(A) = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}A \]

Let \( \beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \)

\[ A = [T]_\beta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \]

\( \Rightarrow \) characteristic polynomial \( (t^2 - t + 1)^2. \)

Note \[ A^2 = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \]

so \( A^2 - A + I = 0. \)

The minimal polynomial is therefore \( t^2 - t + 1. \)
\[ K_{t^2-t+1} = \mathbb{R}^4 \] which has standard basis
\[
\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}
\]

We seek to turn this into a union of disjoint \( T \)-cyclic bases.

Note
\[
\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}
\]
and
\[
\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix}
\]
\[= -\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} \]
Also note

\[
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 \\
1 \\
0 \\
0
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0 \\
-1
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0 \\
-1
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
-1 \\
0 \\
-1
\end{pmatrix}
\]

\[= - \begin{pmatrix}
0 \\
1 \\
0 \\
0
\end{pmatrix} + \begin{pmatrix}
0 \\
0 \\
0 \\
-1
\end{pmatrix}\]

\[\Rightarrow \left\{ \begin{pmatrix}
1 \\
0 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
0 \\
0 \\
-1 \\
0
\end{pmatrix} \right\} \cup \left\{ \begin{pmatrix}
0 \\
1 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
0 \\
0 \\
0 \\
-1
\end{pmatrix} \right\}
\]

are two $T$-cyclic bases whose union gives a rational canonical basis for $K_{t^2-t+1}$.
A rational canonical basis is
\[ \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \right\} \]
with corresponding rational canonical form
\[
\begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 1
\end{pmatrix}
\]
Example

Find the rational canonical form of the matrix

\[ A = \begin{pmatrix}
0 & 2 & 0 & -6 & 2 \\
1 & -2 & 0 & 0 & 2 \\
1 & 0 & 1 & -3 & 2 \\
1 & -2 & 1 & -1 & 2 \\
1 & -4 & 3 & -3 & 4
\end{pmatrix} \]

and a corresponding rational canonical basis.

Hint: the characteristic polynomial is

\[-(t^2 + 2)^2(t - 2).\]