The following sections of the algebra notes will not be examined: 1.1, 2.5, 5.14, 5.15. Another good source of problems are the practice problems on assignments 1–4.

Practice Algebra Exam 1

Question 1.

(a) Find a solution of the equation $883x + 451y = 1$ for integers $x$ and $y$.
(b) Calculate $250 \pmod{100}$.
(c) Prove that $\gcd(a, b) = 1$ if and only if $\gcd(a^2, b^2) = 1$.
(d) Show that 2 is a primitive root mod 81. Hence solve the equation $x^5 \equiv 28 \pmod{81}$.
(e) Solve the following system of simultaneous congruences:
   
   $x \equiv 2 \pmod{3}$
   
   $x \equiv 3 \pmod{5}$
   
   $x \equiv 4 \pmod{7}$

Question 2.

(a) Give the Cayley table for $G = (\mathbb{Z}/24\mathbb{Z})^\times$. It is known that there are only 3 abelian groups of order 8 up to isomorphism: $\mathbb{Z}/8\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Which one is $G$ isomorphic to?
(b) Let $K$ be the set of non-zero real numbers. Define an operation $*$ on $K$ by
   
   $a * b = \begin{cases} 
   ab, & \text{if } a > 0 \\
   a/b, & \text{if } a < 0
   \end{cases}$

   Does $K$ form a group? Explain.
(c) Give the definition of a normal subgroup.

Question 3.

Let $G = \mathbb{R} \setminus \{-1\}$, and define a binary operation $\circ$ on $G$ by $a \circ b = a + b + ab$.

(a) Prove that $(G, \circ)$ is a group.
(b) Find all the elements of order exactly 2.
(c) Let $GL_2(\mathbb{R})$ denote the group of invertible 2 by matrices with real entries under matrix multiplication. Let $T$ denote the subset of upper triangular matrices of the form

   $\begin{pmatrix} 1 & a \\ 0 & 1 + a \end{pmatrix}$, \hspace{1cm} a \in \mathbb{R} \setminus \{-1\}$

   Prove that $T$ is a subgroup of $GL_2(\mathbb{R})$.
(d) Define a function $f: G \to T$ by
   
   $f(a) = \begin{pmatrix} 1 & a \\ 0 & 1 + a \end{pmatrix}$, \hspace{1cm} a \in \mathbb{R} \setminus \{-1\}$

   Prove that $f$ is a group isomorphism.

Challenge Using the previous part (or by some other means), determine all elements in $G$ with finite order.

Question 4. [This may be easier after the linear algebra part of the course.] Let $V$ be a real vector space. Let $\mathcal{R}$ be the set of all linear transformations $V \to V$. If $f, g: V \to V$, define $f + g: V \to V$ by $(f + g)(v) = f(v) + g(v)$ for each $v$.

(a) Prove that $\mathcal{R}$ forms a ring with identity under the addition defined above and with “multiplication” given by composition.
(b) Let $\mathcal{S}$ be the set of all functions $V \to V$ (not just linear functions). Does $\mathcal{S}$ form a ring? Explain.
(c) Which of the following are fields? $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$, $\mathbb{F}_p$, $\mathbb{Z}[x]$, $\mathbb{R}[x]$. 

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Practice Algebra Exam 2

Question 1.
(a) Let \( \varphi \) denote the Euler \( \varphi \) function. Calculate \( \varphi(100) \). (Show your work.)
(b) Prove that \( \varphi(n^2) = n\varphi(n) \) for all natural numbers \( n \).
(c) Find a solution of the equation \( 217x + 101y = 1 \) in integers \( x \) and \( y \), or prove that none exists.
(d) Prove that if \( a \mid b \) and \( b \mid c \) then \( a \mid (2b + 3c) \).
(e) Show that \( 2 \) is a primitive root mod 25. Hence solve the equation \( x^3 \equiv 16 \pmod{25} \).

Question 2. Let \( G \) be a group.
(a) Prove that \( G \) is abelian if and only if \( (ab)^2 = a^2b^2 \) for all \( a, b \in G \).
(b) The centre of \( G \), denoted \( Z(G) \), is defined to be the following set:
\[
Z(G) = \{ x \in G \mid gx = xg \text{ for every } g \in G \}
\]
Prove that \( Z(G) \) is a subgroup of \( G \).
(c) Prove that \( Z(G) \trianglelefteq G \).

Question 3.
(a) Let \( S_3 \) denote the symmetric group on 3 letters. Let
\[
\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.
\]
Show that \( \tau\sigma = \sigma^2\tau \).
(b) Describe all the cyclic subgroups of \( S_3 \). Which of them are normal in \( S_3 \)?
(c) Give the Cayley table for the group \( H = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \) under addition. Is \( H \) isomorphic to \( S_3 \)? Explain.
(d) State and prove Lagrange's theorem for subgroups of a finite group.

Question 4.
(a) Let \( \mathbb{C} \) denote the complex numbers, and \( M_2(\mathbb{R}) \) the ring of 2 by 2 matrices with real entries. Define a map \( f : \mathbb{C} \rightarrow M_2(\mathbb{R}) \) by
\[
f(a + bi) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}
\]
Prove that \( f \) is an injective homomorphism of rings. Is it an isomorphism?
(b) Prove that every finite integral domain is a field. Hint: if \( A \) is an integral domain and \( a \in A \), consider the map \( A \rightarrow A \) sending \( x \mapsto ax \).