MATH 2200: Assignment 6 (Scientific Computing)

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This assignment is worth 10% of the total mark for the course overall. It is due by 4.00pm on Friday October 17th, in the course Assignment Box in the Priestley Building (67).

This assignment is taken from the textbook for this course, Sauer p. 123-125, attached. You are expected to complete the first 6 questions. The marking scheme and additional comments for each question are contained below.

You must hand in a hard-copy of the assignment for marking (assignments on disk, CD, etc. will not be accepted). Please comment your code extensively so it is obvious to the markers what you are trying to do. Uncommented code will lose marks.

1. Hand in: m-file, print-out showing the convergence (or lack thereof) of both the Jacobi and Gauss-Seidel methods, comment on convergence, number of steps required to converge within 6 d.p. (5 marks.)

2. Hand in: Matlab plot showing both the convergent solution from Q1 and the exact solution, output showing the error at $x = 5$ (2 marks.)

3. Note that the size of the matrices in the question is going to get very large- if $k = 11$, then A is a 20480$\times$20480 matrix. If this is taking too long (i.e. longer than about 2-5 minutes) to evaluate on your computer and the error is increasing as you increase n, then you can stop re-running the calculation at a value of $k = 7$. Use comments to justify what you do and why. Hand in: m-file; table showing i) errors at $x = 5$ for each value of n, and ii) condition number for each value of n. Write comments about the trends in the error in your table- why does the error increase with n?(4 marks.)

4. As in Q3, if the evaluation is taking longer than about 2-5 minutes to evaluate and the error is increasing, then stop evaluating at $k = 7$. Hand in: m-file, table showing the solution to the problem for values of k, plot showing the solutions for different values of k, comment on the x value at which the maximum displacement occurs. (3 marks)

5. Evaluate for appropriate values of k as in Q3 and Q4. Hand in: m-file and plot comparing Matlab solution to exact solution, comparison of the condition numbers of the cantilever beam and the pinned beam for the values of n in Step 3. (4 marks)

6. Hand in: m-file, plot showing the displacement for the loaded beam (Q6) and the unloaded beam (Q5) for the values of n in Q3. (2 marks)

Reality Check 2 THE EULER-BERNOULLI BEAM

The Euler-Bernoulli beam is a simple model for bending under stress. Discretization converts the differential equation model into a system of linear equations. Decreasing the discretization size gives larger and larger systems of equations. Surprisingly, the system size is limited by ill-conditioning long before computation time becomes a factor.

The vertical displacement of the beam is represented by a function $y(x)$, where $0 \le x \le L$ along the beam of length L. We will use MKS units in the calculation: meters, kilograms, and seconds. The displacement $y(x)$ satisfies the Euler-Bernoulli equation

$$EIy'''' = f(x), \tag{2.39}$$

where E, the Young's modulus of the material, and I, the area moment of inertia, are constant along the beam. The right-hand side $f(x)$ is the applied load, including the weight of the beam, in force per unit length.

Techniques for discretizing derivatives are found in Chapter 5, where it will be shown that a reasonable approximation for the fourth derivative is

$$y''''(x) \approx \frac{y(x-2h) - 4y(x-h) + 6y(x) - 4y(x+h) + y(x+2h)}{h^4} \tag{2.40}$$

for a small increment h. (See Exercise 5.1.19.) The discretization error of this approximation is proportional to h^2. Our strategy will be to consider the beam as the union of many segments of length h, and to apply the discretized version of the differential equation to each segment.

For a positive integer n, set $h = L/(n+1)$. Consider the evenly-spaced grid $0 = x_0 < x_1 < \cdots < x_n < x_{n+1} = L$, where $h = x_i - x_{i-1}$ for $i = 1, \ldots, n$. Replacing the differential equation (2.39) with the difference approximation (2.40) to get the system of linear equations for the displacements $y_i = y(x_i)$ yields

$$y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2} = \frac{h^4}{EI} f(x_i) \tag{2.41}$$

for $i = 1, \ldots, n$. There are n equations in the n unknowns $y_1, \ldots, y_n$. The coefficient matrix, or structure matrix, will have coefficients from the left-hand side of this equation, altered slightly to take into account the assumptions on the supports at the ends of the beam.

First consider the beam with boundary conditions

$$y(0) = y'(0) = y(L) = y'(L) = 0,$$

called the **pinned beam.** It is fixed at both ends, and its slope at both ends is constrained to be zero. The Euler-Bernoulli equation allows calculation of the sag in the middle.

The end conditions of the pinned beam are handled by using the approximation

$$y''''(x) \approx \frac{12y(x+h) - 6y(x+2h) + \frac{4}{3}y(x+3h)}{h^4}$$

where $x = 0$. The approximation is valid when $y(x) = y'(x) = 0$, as shown in Exercise 5.1.20(a). Therefore, the version of (2.41) that takes the endpoint condition into account for $i = 1$ is

$$12y_1 - 6y_2 + \frac{4}{3}y_3 = \frac{h^4}{EI}f(x_0).$$

The same consideration also applies to the right end of the beam, resulting in the following system of equations for the pinned beam:

$$\begin{bmatrix} 12 & -6 & \frac{4}{3} & & & & & & \\ -4 & 6 & -4 & 1 & & & & & \\ 1 & -4 & 6 & -4 & 1 & & & & \\ & 1 & -4 & 6 & -4 & 1 & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & 1 & -4 & 6 & -4 & 1 & \\ & & & & 1 & -4 & 6 & -4 & 1 \\ & & & & & 1 & -4 & 6 & -4 \\ & & & & & & \frac{4}{3} & -6 & -12 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \\ \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} = \frac{h^4}{EI} \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ \\ \\ \vdots \\ f(x_{n-1}) \\ f(x_n) \end{bmatrix}. \quad (2.42)$$

The structure matrix A is a **banded matrix**, meaning that all entries sufficiently far from the main diagonal are zero. In (2.42) the matrix entries $a_{ij} = 0$, except for $|i - j| \leq 2$. The **bandwidth** of this banded matrix is 5.

Now we move on to specific cases: Consider a solid steel beam of length 10 meters, with depth 5 cm and width 10 cm. The density of steel is approximately 7850 kg/m^3. (A Newton of force is 1 kg-m/sec^2, and the Young's modulus of steel is approximately 2×10^{11} Pascals, or Newton/m^2.) The area moment of inertia I around its center of mass is $bh^3/12$, where d is the depth and b is the width of the beam.

We begin by calculating the displacement of the beam with no payload, so that $f(x)$ represents only the weight of the beam itself. Fortunately, there is a closed form solution of (2.39) in that case, so that we can compare the result of our computation for accuracy. The weight f of the beam per meter of length is the constant $7850bd$ kg/m multiplied by the acceleration of gravity 9.81 m/sec^2. The reader should check that the units match on both sides of (2.39).

Suggested activities:

1. Write a MATLAB program to define the structure matrix A in (2.42), using MATLAB's `spdiags` command. Solve the system for the displacements y_i with $n = 10$, trying the Jacobi and Gauss-Seidel Methods. Which one converges? How many steps are required to converge within six decimal places?
2. Plot the solution from Step 1 against the correct solution $y(x) = fx^2(L - x)^2/(24EI)$, where $f = f(x)$ is the previously defined constant. Note that the displacements of the beam will be nonnegative numbers, so a realistic plot of a sagging beam will reverse the sign of the displacements. Compute the error at the midpoint of the beam, $x = 5$ meters.

3. Rerun the calculation in Step 1 for $n = 10 \cdot 2^k$, where $k = 1, \ldots, 11$. Make a table of the errors at $x = 5$ for each n. Why does the error begin to increase with n? You may want to make an accompanying table of the condition number of A to help answer the last question.
4. Now add a 1000 kg payload to the beam, covering the space from $x = 6$ to $x = 7$. You must add 1000 kg/m$^2 \cdot 9.81$ m/sec^2 to $f(x_i)$ for all $6 \le x_i \le 7$, and solve the problem again with the values of n from Step 3. Plot the solution. Where along the beam does the maximum displacement occur?
5. Now unhook one end of the beam. A **cantilever beam**, with free boundary conditions at the right end, satisfies the conditions

$$y(0) = y'(0) = y''(L) = y'''(L) = 0.$$

The cantilever beam system of equations (see Exercise 5.1.20 for a derivation) is

$$\begin{bmatrix} 12 & -6 & \frac{4}{3} & & & & & & \\ -4 & 6 & -4 & 1 & & & & & \\ 1 & -4 & 6 & -4 & 1 & & & & \\ & 1 & -4 & 6 & -4 & 1 & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & 1 & -4 & 6 & -4 & 1 & \\ & & & & 1 & -4 & 6 & -4 & 1 \\ & & & & & 1 & -\frac{93}{25} & \frac{111}{25} & -\frac{43}{25} \\ & & & & & & \frac{12}{25} & -\frac{24}{25} & \frac{12}{25} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \\ \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} = \frac{h^4}{EI} \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ \\ \\ \vdots \\ f(x_{n-1}) \\ f(x_n) \end{bmatrix}. \quad (2.43)$$

Repeat Step 3 for the unloaded cantilever beam, comparing with the correct solution $y(x) = (f/24EI)x^2(x^2 - 4Lx + 6L^2)$. Are the condition numbers of the structure matrix better or worse than for the pinned beam?
6. Apply the load to the cantilever beam, as in Step 4, and recalculate. Plot the unloaded beam in the same plot as the loaded beam.
7. The following are some ideas for further exploration: Is the beam stronger or weaker if the rectangular cross-section is changed to $h = 10$ cm and $b = 5$ cm? or if the cross-section is circular or annular with the same area as the rectangle? (The area moment of inertia for a circular cross-section of radius r is $I = \pi r^4/4$, and for an annular cross-section with inner radius r_1 and outer radius r_2 is $I = \pi(r_2^4 - r_1^4)/4$.) Find out the area moment of inertia for I-beams, for example. The Young's modulus for different materials are also tabulated and available.

The Euler-Bernoulli beam is a relatively simple, classical model. More recent models, such as the Timoshenko beam, take into account more general bending—where the beam's cross-section may not be perpendicular to its main axis—and are often more accurate.