

Weighting: 10%.

Due Date: 4.00pm on 26 Sept 2008, in the course Assignment Box in the Priestley Building (67).

Please hand in a hard-copy of the assignment (assignments on disk, CD, etc. will not be accepted). For the problems referring to the explicit trapezoid method and the order four Runge-Kutta method, you need to write your own code in Matlab. Other problems require that you use the *ode45* and *ode23s* methods, which are in-build solvers in Matlab.

1. (*2 points*) The Lorentz equations correspond to the first simplified version of convection movement in the atmosphere:

$$x' = s(y - x) \tag{1}$$

$$y' = x(r - z) - y$$

$$z' = xy - bz \tag{2}$$

For this problem take $s = 10$, $r = 28$ and $b = 8/3$. Using the trapezoid method with step size $h = 0.001$, calculate the trajectory $Y_a(t) = (x(t), y(t), z(t))$ with initial condition $Y_a(0) = (5, 5, 5)$, for $t < 20$. Compare the solution to the trajectory $Y_b(t)$ obtained with initial condition $Y_b(0) = (5 + \Delta, 5, 5)$, with $\Delta = 10^{-5}$. Calculate the difference between both trajectories:

$$d(t) = \|Y_a(t) - Y_b(t)\|. \tag{3}$$

Plot $d(t)$ versus time using the *semilogy* matlab function. Is the difference between both trajectories inherent from the equations? or it is a numerical artifact? (Hint: Calculate $d(t)$ using trapezoid method and four order Runge-Kutta method and compare the results.)

2. (*4 points*) The Lotka-Volterra equations are often used to simulate interaction between populations. A particular case of these equations describes the variation of two competing species, let's say, cane toads and green tree frogs:

$$x' = ax(1 - x) - bxy \tag{4}$$

$$y' = cy(1 - y) - dxy$$

where x and y is the population of the species normalized by the carrying capacity of the ecosystem. a, b, c and d are positive constants.

(a) (*2 point*) Determine analytically the equilibrium points of the system. Set the parameters $a = 2$, $b = 3$, $c = 1$ and $d = 2$. Then use the solver *ode45* of Matlab to calculate the trajectories in the phase space (x, y) for different initial conditions ($0 < x(0) < 1$ and $0 < y(0) < 1$). From the numerical results discuss whether the equilibrium points of the system are stable.

(b) (2 point) Refine the trajectories in the phase space to determine which initial populations lead one or the other species eventually dies out.

3. (4 points) The Van der Pol equation constitutes a simplified equation to model electrical activity of the heart:

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = 0 \quad (5)$$

(a) (1 point) Study the behavior of the system when $\mu \ll 1$: Take $\mu = 0.1$ and use the four order Runge-Kutta method with step $h = 0.1$ to solve numerically this equation in the interval $0 < t < 200$ using different initial conditions: $x(0) = 1, 2, \dots, 5$ and $x'(0) = 0$. Plot $x(t)$ versus time as well as the trajectories in the phase space $(x(t), x'(t))$ of all solutions. In which extent is the behavior of the system similar to the harmonic oscillator?

(b) (1 point) The extreme case when $\mu \gg 1$ is an example of a *stiff equation*. Take $\mu = 100$ and initial condition $x(0) = 1, x'(0) = 0$. Solve the differential equation in the interval $0 < t < 200$ using three different solvers: order four Runge-Kutta with steps $h = 0.005$, *ode45* and *ode23s*. Compare $y(t)$ versus time for the three cases. Estimate the time spent by each solver using the *tic* and *toc* Matlab functions. Calculate also the number of time steps used by each solver by using *length(t)*, where t is the time vector returned by the solver. Based on these results discuss which method is more suitable to solve this stiff equation.

(c) (2 points) Use a suitable solver from matlab to study the limit behavior (as $t \rightarrow \infty$) for an intermediate value of μ . You will observe that the system evolves towards a limit state cycle (or *attractor*) in the phase state. Plot the attractors in the phase space for different values of μ . Does this limit state depend on the initial conditions? (Rather than with a formal mathematical demonstration, justify your response with a sketch of the phase portrait of the Van der Pol equation.)