MATH 2200 (Semester 2, 2008), Assignment 1

A/Prof Geoffrey Goodhill

This assignment is worth 10% of the total mark for the course. It is due by 4.00pm on Friday August 8th, in the Math2200 Assignment Box in the Priestley Building (67).

The assignment is most of Reality Check 1 in the required text for the course, Numerical Analysis by Timothy Sauer (pp71-73). A copy is attached for your reference. You are expected to complete the first 6 questions. The marking scheme and some additional comments for each question are contained below. There are total of 20 marks; each corresponds to 0.5% of the total course mark.

As stated in the Course Profile, late assignments will be penalized by 1% of the total course mark per day or part-day late. If you do hand in late please alert the tutors so that they know to pick it up from the assignment box.

You must hand in hard-copy of your solutions for marking (no disks, CDs etc). Please comment your code extensively so it is obvious to the markers what you are trying to do. Uncommented code will lose marks. Note that almost all the algebra needed for this assignment is given to you, so if you find yourself needing to do large algebraic re-arrangements etc you may have gone down the wrong path.

1. Note that $x_1 = 4, x_2 = 0, y_2 = 4$. Hand in: m-file for $f(\theta)$, print-out of matlab sessions showing output at $\theta = \pm\frac{\pi}{4}$. (4 marks)

2. Include in your plot the line $y = 0$ so the x-intercepts are easy to view. Do this for all plots of $f(\theta)$ in this assignment. Hand in: matlab commands, plot. (2 marks)

3. Hand in: any relevant m-files, and matlab sessions used to produce output, plots. (4 marks)

4. You may not use MATLAB built-in solvers (such as fsolve) for this question. Your answer needs to includes plots of the Stewart platform at all 4 poses (in addition to the plot of $f(\theta)$) and the verification of p_1, p_2, p_3 requested. You must also show the 4 solutions of $f(\theta)$, correct to at least 4 decimal places. Hand in: m-files, matlab sessions, plots. (6 marks)

5. Include a plot of $f(\theta)$ to show there are 6 solutions. Use your numerical method to find the 6 solutions correct to at least 4 decimal places. Hand in: any relevant m-files, matlab sessions, plot. (2 marks)

6. Include a plot of $f(\theta)$ to show there are 2 solutions and solve numerically to find both solutions correct to at least 4 decimal place. Hand in: any relevant m-files, matlab sessions, plot. (2 marks)

5. In Exercise 6, you were asked what the outcome of the Bisection Method would be for $f(x) = 1/x$ on the interval $[-2, 1]$. Now compare that result with applying `fzero` to the problem.

6. What happens if `fzero` is asked to find the root of $f(x) = x^2$ near 1 (do not use a bracketing interval)? Explain the result. (b) Apply the same question to $f(x) = 1 + \cos x$ near -1.

Reality Check 1 KINEMATICS OF THE STEWART PLATFORM

A Stewart platform consists of six variable length struts, or prismatic joints, supporting a payload. Prismatic joints operate by changing the length of the strut, usually pneumatically or hydraulically. As a six-degree-of-freedom robot, the Stewart platform can be placed at any point and inclination in three-dimensional space that is within its reach.

To simplify matters, the project concerns a two-dimensional version of the Stewart platform. It will model a manipulator composed of a triangular platform in a fixed plane controlled by three struts, as shown in Figure 1.14. The inner triangle represents the Planar Stewart Platform whose dimensions are defined by the three lengths L_1, L_2, and L_3. Let γ denote the angle across from side L_1. The position of the platform is controlled by the three numbers p_1, p_2, and p_3, the variable lengths of the three struts.

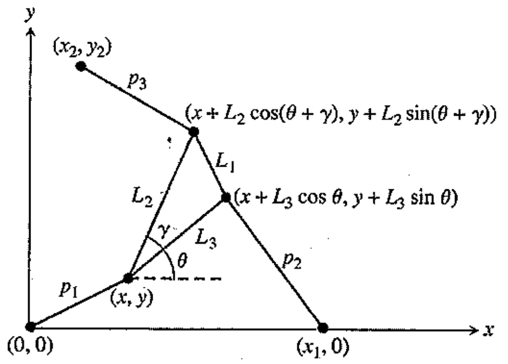

Figure 1.14 Schematic of Planar Stewart Platform. The forward kinematics problem is to use the lengths p_1, p_2, p_3 to determine the unknowns x, y, θ.

Finding the position of the platform, given the three strut lengths, is called the forward, or direct, kinematics problem for this manipulator. Namely, the problem is to compute (x, y) and θ for each given p_1, p_2, p_3. Since there are three degrees of freedom, it is natural to expect three numbers to specify the position. For motion planning, it is important to solve this problem as fast as possible, often in real time. Unfortunately, no closed form solution of the Planar Stewart Platform forward kinematics problem is known.

The best current methods involve reducing the geometry of Figure 1.14 to a single equation and solving it by using one of the solvers explained in this chapter. Your job is to complete the derivation of this equation and write code to carry out its solution.

Simple trigonometry applied to Figure 1.16 implies the following three equations:

$$\begin{aligned} p_1^2 &= x^2 + y^2 \\ p_2^2 &= (x + A_2)^2 + (y + B_2)^2 \\ p_3^2 &= (x + A_3)^2 + (y + B_3)^2. \end{aligned} \tag{1.38}$$

In these equations,

$$\begin{aligned} A_2 &= L_3 \cos\theta - x_1 \\ B_2 &= L_3 \sin\theta \\ A_3 &= L_2 \cos(\theta + \gamma) - x_2 = L_2[\cos\theta\cos\gamma - \sin\theta\sin\gamma] - x_2 \\ B_3 &= L_2 \sin(\theta + \gamma) - y_2 = L_2[\cos\theta\sin\gamma + \sin\theta\cos\gamma] - y_2. \end{aligned}$$

Note that (1.38) solves the inverse kinematics problem of the Planar Stewart Platform, which is to find p_1, p_2, p_3, given x, y, θ. Your goal is to solve the forward problem, namely, to find x, y, θ, given p_1, p_2, p_3.

Multiplying out the last two equations of (1.38) and using the first yields

$$\begin{aligned} p_2^2 &= x^2 + y^2 + 2A_2x + 2B_2y + A_2^2 + B_2^2 = p_1^2 + 2A_2x + 2B_2y + A_2^2 + B_2^2 \\ p_3^2 &= x^2 + y^2 + 2A_3x + 2B_3y + A_3^2 + B_3^2 = p_1^2 + 2A_3x + 2B_3y + A_3^2 + B_3^2, \end{aligned}$$

which can be solved for x and y as

$$\begin{aligned} x &= \frac{N_1}{D} = \frac{B_3(p_2^2 - p_1^2 - A_2^2 - B_2^2) - B_2(p_3^2 - p_1^2 - A_3^2 - B_3^2)}{2(A_2B_3 - B_2A_3)} \\ y &= \frac{N_2}{D} = \frac{-A_3(p_2^2 - p_1^2 - A_2^2 - B_2^2) + A_2(p_3^2 - p_1^2 - A_3^2 - B_3^2)}{2(A_2B_3 - B_2A_3)}, \end{aligned} \tag{1.39}$$

as long as $D = 2(A_2B_3 - B_2A_3) \neq 0$.

Substituting these expressions for x and y into the first equation of (1.38), and multiplying through by D^2, yields one equation, namely,

$$f = N_1^2 + N_2^2 - p_1^2 D^2 = 0 \tag{1.40}$$

in the single unknown θ. (Recall that $p_1, p_2, p_3, L_1, L_2, L_3, \gamma, x_1, x_2, y_2$ are known.) If the roots of $f(\theta)$ can be found, the corresponding x and y values follow immediately from (1.39).

Note that $f(\theta)$ is a polynomial in $\sin\theta$ and $\cos\theta$, so, given any root θ, there are other roots $\theta + 2\pi k$ that are equivalent for the platform. For that reason, we can restrict attention to θ in $[-\pi, \pi]$. It can be shown that $f(\theta)$ has at most six roots in that interval.

Suggested activities:

1. Write a MATLAB function file for $f(\theta)$. The parameters $L_1, L_2, L_3, \gamma, x_1, x_2, y_2$ are fixed constants, and the strut lengths p_1, p_2, p_3 will be known for a given pose. To test your code, set the parameters $L_1 = 2, L_2 = L_3 = \sqrt{2}, \gamma = \pi/2, p_1 = p_2 = p_3 = \sqrt{5}$ from

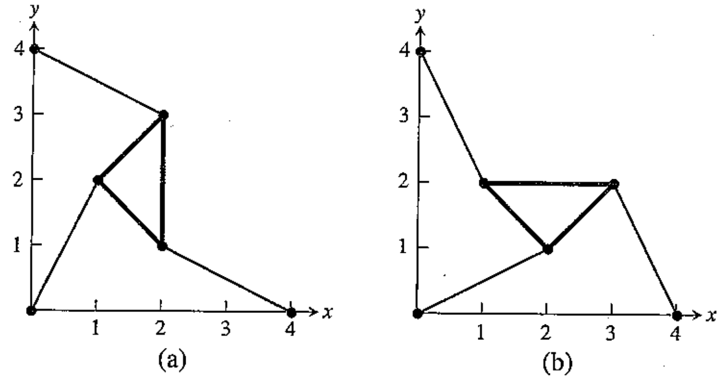

Figure 1.15 Two poses of the Planar Stewart Platform with identical arm lengths. Each pose corresponds to a solution of (1.38) with strut lengths $p_1 = p_2 = p_3 = \sqrt{5}$. The shape of the triangle is defined by $L_1 = 2$, $L_2 = L_3 = \sqrt{2}$, $\gamma = \pi/2$.

Figure 1.15. Then, substituting $\theta = -\pi/4$ or $\theta = \pi/4$, corresponding to Figures 1.15(a) or 1.15(b), respectively, should make $f(\theta) = 0$.

2. Plot $f(\theta)$ on $[-\pi, \pi]$. There should be roots at $\pm\pi/4$.
3. Reproduce Figure 1.15. The MATLAB commands

```
>> plot([u1 u2 u3 u1],[v1 v2 v3 v1],'r'); hold on
>> plot([0 x1 x2],[0 0 y2],'bo')
```

will plot a red triangle with vertices `(u1,v1)`, `(u2,v2)`, `(u3,v3)` and place small circles at the strut anchor points `(0,0)`, `(0,x1)`, `(x2,y2)`. In addition, draw the struts.

4. Solve the forward kinematics problem for the planar Stewart platform specified by $x_1 = 5$, $(x_2, y_2) = (0, 6)$, $L_1 = L_3 = 3$, $L_2 = 3\sqrt{2}$, $\gamma = \pi/4$, $p_1 = p_2 = 5$, $p_3 = 3$. Begin by plotting $f(\theta)$. Use an equation solver from Chapter 1 to find all four poses, and plot them. Check your answers by verifying that p_1, p_2, p_3 are the lengths of the struts in your plot.
5. Change strut length to $p_2 = 7$ and re-solve the problem. For these parameters, there are six poses.
6. Find a strut length p_2, with the rest of the parameters as in Step 4, for which there are only two poses.
7. Calculate the intervals in p_2, with the rest of the parameters as in Step 4, for which there are 0, 2, 4, and 6 poses, respectively.
8. Derive or look up the equations representing the forward kinematics of the three-dimensional, six-degrees-of-freedom Stewart platform. Write a MATLAB program and demonstrate its use to solve the forward kinematics. See [4] for a good introduction to prismatic robot arms and platforms.