1.* Suppose that \( \rho(r, t) = At^2r^2 \) is the density of some conserved substance in the region \( a < r < b \) of 3-D space. Here \( A \), \( a \) and \( b \) are positive constants, and \( r = \sqrt{x^2 + y^2 + z^2} \). Calculate

\[
Q(t) = \int_a^b 4\pi r^2 \rho(r, t) \, dr,
\]

the quantity of substance in the region at time \( t \).

Find the most general flux vector of the form \( \vec{J}(x, y, z, t) = f(r, t)\vec{r} \), where \( \vec{r} = xi + yj + zk \), such that \( \rho \) and \( \vec{J} \) satisfy the conservation equation

\[
\frac{\partial \rho}{\partial t} + \text{div}\vec{J} = 0.
\]

Show that if \( \vec{J} \cdot \vec{r} = Bt \) when \( r = a \), with \( B \) constant, then

\[
f(r, t) = \frac{aBt}{r^3} - \frac{2At(r^5 - a^5)}{(5r^3)}.
\]

How much substance flows into the region per unit time over the surface \( r = a \)? How much over \( r = b \)?

Check that the total amount flowing into the region per unit time matches \( dQ(t)/dt \).

Show that \( \vec{J} = \text{grad}\Phi \), where

\[
\Phi(r, t) = -\frac{aBt}{r} - \frac{Atr^4}{10} - \frac{2Ata^5}{(5r)}
\]

and hence or otherwise show that \( \text{curl}\vec{J} = \vec{0} \).
Practice Problems:

Problem Set K9.7 p.409 Numbers 16, 18
Problem Set K9.8 p.413 Numbers 4, 8
Problem Set K9.9 p.416 Numbers 2, 6

Solutions to the starred problem to be handed in by 5pm on Tuesday, October 3, in appropriate box on Level 3, Mathematics Bldg 67. Don’t forget a cover sheet!