(5.1) \[ y'(t) = A y(t) \]

\[ A = \begin{bmatrix}
\frac{3}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{3}{2}
\end{bmatrix} \]

**Eigenvalues:** \( \lambda_1 = 1 \), \( \lambda_2 = 2 \) (see p. 4:11)

**Eigenvectors:** \( \lambda_1 = 1 \):

\[ (A - \lambda I)x = 0 \Rightarrow \begin{bmatrix}
\frac{3}{2} - 1 & \frac{1}{2} \\
\frac{1}{2} & \frac{3}{2} - 1
\end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

\[ \Rightarrow \frac{1}{2}u + \frac{1}{2}v = 0 \quad \text{and} \quad \frac{1}{2}u + \frac{1}{2}v = 0 \]

\[ \Rightarrow v = -u \]

Choosing \( u = 1 \), we get \( x^{(1)} = (1) \)

\( \lambda_2 = 2 \):

\[ (A - \lambda I)x = 0 \Rightarrow \begin{bmatrix}
\frac{3}{2} - 2 & \frac{1}{2} \\
\frac{1}{2} & \frac{3}{2} - 2
\end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

\[ \Rightarrow -\frac{1}{2}u + \frac{1}{2}v = 0 \quad \text{and} \quad \frac{1}{2}u - \frac{1}{2}v = 0 \]

\[ \Rightarrow v = u \]

Choosing \( u = 1 \), we get \( x^{(2)} = (1) \)
Now each of \( z^{(1)} e^t \) and \( z^{(2)} e^{2t} \) is a solution.

General solution of (5.1) is:

\[
y(t) = A z^{(1)} e^t + B z^{(2)} e^{2t}
\]

\[
= A \left( \frac{1}{1} \right) e^t + B \left( \frac{1}{1} \right) e^{2t}
\]

Each choice of ICs, or equivalently, each choice of \( A \) and \( B \), defines a trajectory in the \( y, y_2 \)-plane.

Let's look at some of them!
First: Suppose $B=0$, $A>0$

$\Rightarrow y(t) = A \left( \begin{array}{c} 1 \\ -1 \end{array} \right) e^t$

$\Rightarrow y_1(t) = Ae^t$ ($>0$), $y_2(t) = -Ae^t$ ($<0$)

Choosing different (positive) $A$s corresponds to starting at different places on red line at $t=0$. 
For $B = 0$, $A < 0$, we have $y_t = -y_2$, $y_1 < 0$, $y_2 > 0$

Similarly, for $A = 0$, $B > 0$ we get

$y_t = y_2$, $y_1 > 0$, $y_2 > 0$

and for $A = 0$, $B < 0$ we get

$y_t = y_2$, $y_1 < 0$, $y_2 < 0$
Now we have so far 5 trajectories:

Note directions

Note that they do not cross. They meet only at \( t = -\infty \).

To get further info. about the phase-portrait, imagine a trajectory, and think of \( y_2(y_1) \) on it. The slope of this curve is

\[
\frac{dy_2}{dy_1} = \frac{dy_2/dt}{dy_1/dt} \quad \text{(Chain rule!)}
\]

\[
= \frac{\frac{1}{2}y_1 + \frac{3}{2}y_2}{\frac{3}{2}y_1 + \frac{1}{2}y_2} = \frac{y_1 + 3y_2}{3y_1 + y_2}
\]
It follows that
\[ \frac{dy_1}{dy} = 0 \] wherever \( y_1 = -\frac{1}{2} y \).
\[ \frac{dy_1}{dy} = \pm \infty \] wherever \( y_0 = -2y_1 \).

So:

- Every trajectory is vertical when and if it cuts this line \( y_0 = -2y_1 \).
- Every trajectory is horizontal when and if it cuts this line \( y_0 = -\frac{1}{2} y \).
How find directions (of arrows)?

Need info about \( \frac{dy}{dt} \) (velocity vector) — how things change with time.

So, look again at our system:

\[
\frac{dy_1}{dt} = \frac{3}{2} y_1 + \frac{1}{2} y_2
\]
\[
\frac{dy_2}{dt} = -y_1 + \frac{3}{2} y_2
\]

Where \( y_2 = -\frac{1}{3} y_1 \), we have

\[
\frac{dy_1}{dt} = \frac{3}{2} y_1 + \frac{1}{2} \left(-\frac{1}{3} y_1 \right) = \frac{7}{6} y_1
\]

and so

\[
\frac{dy_1}{dt} > 0 \text{ if } y_2 = -\frac{1}{3} y_1 \text{ and } y_1 > 0
\]
\[
\frac{dy_1}{dt} < 0 \text{ if } y_2 = -\frac{1}{3} y_1 \text{ and } y_1 < 0
\]

So:

\[
\begin{array}{c}
\text{y}_2 < 0 \\
\text{y}_1 > 0
\end{array}
\]
Similarly, where \( y_2 = -3y_1 \),

\[
\frac{dy_2}{dt} = \frac{1}{2} y_1 + \frac{3}{2} (-3y_1) = -4y_1
\]

and so,

\[
\frac{dy_2}{dt} < 0 \text{ if } y_2 = -3y_1 \text{ and } y_1 > 0
\]

\[
\frac{dy_2}{dt} > 0 \text{ if } y_2 = -3y_1 \text{ and } y_1 < 0
\]

So:

\( y < 0 \)  \( y > 0 \)
Can get further info. by considering trajectories where they cross other straight lines.

For example, where \( y_1 = 0 \), we have
\[
\frac{dy_2}{dy_1} = \frac{y_1 + 3y_2}{3y_1 + y_2} = \frac{3y_2}{y_2} = 3 \quad \text{if} \quad y_2 > 0
\]
\[
\frac{dy_1}{dt} = \frac{2y_1 + 4y_2}{4y_1 + 2y_2} = \frac{2y_2}{2y_2} = 1 \quad \text{if} \quad y_2 > 0
\]
Similarly, where \( y_2 = 0 \), we have

\[
\frac{dy_1}{dt} = \frac{y_1 + 3y_2}{2y_1 + y_2} = \frac{\frac{1}{3}y_1}{y_1} = \frac{1}{3} \quad (\text{as } y_2 = 0)
\]

\[
\frac{dy_1}{dt} = \frac{y_1 + 3y_2}{2y_1 + y_2} = \frac{\frac{1}{2}y_1}{y_1} = \frac{1}{2} \quad \text{if } y_1 > 0
\]

\[
\frac{dy_1}{dt} = \frac{y_1 + 3y_2}{2y_1 + y_2} = \frac{\frac{1}{3}y_1}{y_1} = \frac{1}{3} \quad \text{if } y_1 < 0
\]
Lecture summary:

1) Improper node: A has two real eigenvalues with same sign.

2) \( y = 0 \) (origin) is a trajectory.

3) Know how to find 1st trajectories radiating from origin.

4) Know how to find \( \frac{dy}{dx} \) and use to get info. about trajectories.

5) Know how to use \( \frac{dy}{dx} \) (or \( \frac{dx}{dy} \)) to get info. about directions of arrows.

K pp. 141 - 144
140 - 142