There exists a unique solution of the IVP

\[ y'(t) = F(t, y(t)) \quad y(0) = k \]

provided the components \( f_1, f_2, \ldots, f_n \) of \( F \) are 'suitably smooth' functions.

(\textit{K, p.137 Theorem 1})

Things go wrong if \( F \) not 'suitably smooth'.
Exercise: \[
\begin{aligned}
y_1' &= \sqrt[4]{\frac{1}{2}(y_1+2)(y_2-1)} \\
y_2' &= 2\sqrt{1 + y_1 + 2}
\end{aligned}
\]

ICs: \( y_1(0) = -2 \), \( y_2(0) = 1 \)

There exist (at least) two solutions:

1) \( y_1(t) = -2 \), \( y_2(t) = 1 \), \(-\infty < t < \infty\)

2) \( y_1(t) = \begin{cases} 
\frac{t^2}{2} - 2, & t \geq 0 \\
-\frac{t^2}{2} - 2, & t < 0 
\end{cases} \)
\( y_2(t) = \begin{cases} 
\frac{t^2}{2} + 1, & t \geq 0 \\
-\frac{t^2}{2} + 1, & t < 0 
\end{cases} \)

Check!
\[ \begin{align*}
\text{Ex: } & \begin{cases} 
    y_1' = y_2 \\ 
    y_2' = \frac{6y_2}{t^2} 
\end{cases} \quad \text{ICs: } y(0) = 6, \ y_2(0) = 6 \\
\end{align*} \]

No solution exists.

[General solution of system is]
\[ y_1(t) = \alpha t^2 + \beta t^{-2}, \quad y_2(t) = 3\alpha t^2 - 2\beta t^{-3} \]

\[ \begin{align*}
\text{Ex: } & \begin{cases} 
    y_1' = y_2 \\ 
    y_2' = (y_1)^2 + e^t 
\end{cases} \quad \text{ICs: } y(0) = 0, \ y_2(0) = 0 
\end{align*} \]

Theorem shows that there exists a unique solution in this case. Unfortunately, it cannot be expressed in terms of known functions.
The simplest systems of ODEs are linear

For \( n=2 \):

\[
\begin{align*}
y_1'(t) &= a_{11}(t)y_1(t) + a_{12}(t)y_2(t) + g_1(t) \\
y_2'(t) &= a_{21}(t)y_1(t) + a_{22}(t)y_2(t) + g_2(t)
\end{align*}
\]

Two coupled, first-order, linear ODEs

Here \( a_{11}(t), a_{12}(t), a_{21}(t), a_{22}(t), g_1(t), g_2(t) \) are given functions.

Can write as

\[ y'(t) = A(t)y(t) + g(t) \]

\[
A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}, \quad y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \quad g(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}
\]

given (known) unknown given (known)
[For general \( n \), \( y(t) \) would be an \( n \)-vector, \( A(t) \) would be \( n \times n \), \( g(t) \) would be an \( n \)-vector, \( \text{const} \).]

For a linear system with \( I \), \( y(t_0) = y_0 \), there exists a unique solution if all the \( \text{adj}(t) \) and \( g(t) \) are continuous (at and near \( t_0 \)). (Kp. 138 Theorem 2).

We shall mainly deal with linear systems

\[
y'(t) = A(t)y(t) + g(t),
\]

mainly with \( n = 2 \), mainly in the homogeneous case \( g(t) = 0 \), and mainly in the case \( A = \text{const.} 2 \times 2 \) matrix!

[Then IVP always has a unique solution!]
(4.1) \[ y'(t) = A\dot{y}(t), \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \]

(a_{12}, a_{21} \text{ not necessarily equal to 0, i now!})

ICs: \[ y(0) = y_0 = (y^{10}_0) \]

Recall for ODE \[ y''(t) = F(t, y(t), y'(t)) \]
to visualize solution we draw graph of (or get computer to draw graph of) \( y(t) \) versus \( t \).

What to do to visualize solution of (4.1)?

Could draw graphs of \( y(t) v. t \) and \( y(t)v.t \).
A more interesting thing is to consider \((y_1(t), y_2(t))\) as \textit{coordinates} of a point \(P\) moving with time in the \(y_1,y_2\)-plane, starting at \((y_{10}, y_{20})\) at time \(t_0\):

![Diagram of a trajectory in the \(y_1,y_2\)-plane with arrows indicating the direction of increasing time \(t\).

Arrows on trajectory of \(P\) indicate direction of increasing time \(t\).

(See \(y(t)\) is \textit{position vector} of \(P\) at time \(t\).)
If we consider many different ICs, leading to many different trajectories, we build up a phase-portrait of the system of ODEs:

The $y,y_2$-plane is usually called the phase-plane.
Note: Two trajectories cannot cross (and a trajectory cannot cross itself) at a finite value of $t$.

**pf:**

Suppose

\[ y_A(t_a) = y_P. \]

\[ y_B(t_b) = y_P. \]

Let

\[ y'(t) = y_A(t+t_a) - y_B(t+t_b) \]

Then

\[ y'(t) = Ay(t) \quad \text{and} \quad y(0) = 0. \]

Check!

But one solution of these equations is:

By uniqueness theorem, \[ y(t) = 0 \quad \text{if} \quad t = 0 \]

\[ \Rightarrow y_A(t+t_a) = y(t+t_b) \]

\[ \Rightarrow \text{trajectories look same near } P_0 \quad \text{do not cross.} \]
For a 2-component system

\[(4.1) \quad y'(t) = A \cdot y(t), \quad A = \text{const,} \quad 2 \times 2\]

there are 6 types of phase-portrait that can arise. We will now go through them one-by-one.

Note firstly: \( y(t) = 0 \) is always a solution, however \( A \) looks.

The **trivial solution**

The unique solution of (4.1) with ICs \( y(t_0) = 0 \), any \( t_0 \).

This trajectory is easy to plot!

\[
\begin{align*}
\text{traj.} \\
\end{align*}
\]
Type 1: Improper Node

(A has two real eigenvalues with same sign.)

Ex: \[ A = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} \]

(1) \[ y'(t) = A y(t) \iff \begin{cases} y_1' = \frac{3}{4} y_1 + \frac{1}{4} y_2 \\ y_2' = \frac{1}{4} y_1 + \frac{3}{4} y_2 \end{cases} \]

Try \[ y(t) = x e^{\lambda t}, \quad x = \begin{pmatrix} u \\ v \end{pmatrix} \text{ (const.)} \]

\[ \Rightarrow y'(t) = \lambda x e^{\lambda t} \]

(2) \[ A x = \lambda x \]

\[ \Rightarrow (A - \lambda I)x = 0 \]

\[ \Rightarrow Eigenvalue \text{ condition:} \]

0 = \det(A - \lambda I) = \begin{vmatrix} \frac{3}{4} - \lambda & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} - \lambda \end{vmatrix}

= (\frac{3}{4} - \lambda)^2 - \frac{1}{4} = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1)

Eigenvalues: \(\lambda_1 = 1, \quad \lambda_2 = 2\)
Summary:

1) Understand ideas of existence and uniqueness (K pp 137, 138, Theorems 1, 2) - no need to know details.

2) Understand idea of phase-plane and trajectories (→ phase-portrait)

3) Trajectories never cross.

4) \( y(t) = 0 \) is a solution of \( y'(t) = Ay(t) \)

Whatever \( A \) is, trajectory is the point at origin 0 in \( y_1, y_2 \)-plane.

K pp 137, 138, \( f(x,y,t) = 0, \alpha^1, \alpha^2 \)