EX: Semi-infinite slab

PDE: \( u_t(x,t) = c^2 u_{xx}(x,t) \), \( 0 < x < \infty, \ t > 0 \)

IC: \( u(x,0) = F(x) = \begin{cases} u_0, & 0 < x < l \\ 0, & x > l \end{cases} \)

BC: \( u_x(0,t) = 0, \ t > 0 \)

F(x), \ 0 < x < \infty \rightarrow f(x) = \begin{cases} F(x), & 0 < x < \infty \\ F(-x), & -\infty < x < 0 \end{cases}

Now have Problem 2 on Tut. Sheet 9.

Solution:
\[
\begin{align*}
  u(x,t) &= \frac{1}{2} u_0 \left[ \text{erf} \left( \frac{x+l}{\sqrt{4ct}} \right) - \text{erf} \left( \frac{x-l}{\sqrt{4ct}} \right) \right] \\
&\quad \text{but only for} \ x > 0, \ t > 0.
\end{align*}
\]
Graphs of \( \pm \text{erf} \left( \frac{x + a}{\sqrt{4Dt}} \right) \) and

\[ c(x, t) = \pm C_0 \left[ \text{erf} \left( \frac{x + a}{\sqrt{4Dt}} \right) - \text{erf} \left( \frac{x - a}{\sqrt{4Dt}} \right) \right]. \]
Conduction of heat sometimes called 'thermal diffusion.' The same PDE governs process of molecular diffusion.

Now \( u(x,t) \rightarrow c(x,t) \) concentration of diffusate

\[ c^2 \rightarrow D \text{ coeff. of diffusion} \]

\[ u_t = c^2 u_{xx} \rightarrow c_t(x,t) = D c_{xx}(x,t) \]

1-D Diffusion Equation.

Typical value of \( D \approx 10^{-5} \text{cm}^2/\text{sec} \)

\[ (\rightarrow \text{ over length scales } \approx 5 \text{cm, typical diffusion times } \approx \frac{25}{10^{-5}} \text{sec} \approx 30 \text{days}; \]

over length scales \( \approx 10^{-4} \text{cm, typical diffusion times } \approx \frac{10^{-8}}{10^{-5}} \text{sec} \approx 10^{-3} \text{sec.} \]

- very important transport process for cells.

Before leaving heat conduction/diffusion, consider:
Temperature waves in the earth
(The earthworm's Christmas problem)

We consider variation of temperature in soil as a result of periodic variation of temperature at surface (a) daily or (b) yearly.

Model as semi-infinite region with sinusoidal BC:

\[ u = u_0 \cos(\omega t) \]

We have

\[ \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \quad \text{for} \quad 0 < x < \infty, -\infty < t < \infty \] (31.1)

BC: \[ u(0,t) = u_0 \cos(\omega t) \] (31.2)

IC: irrelevant - we are interested in steady-state behaviour.
Solution should be periodic in time with angular frequency $\omega$.

We consider the PDE for complex-valued $u$, with the complex BC $u(0, t) = u_0 e^{i\omega t}$.

We look for a complex solution in the form $u(x, t) = X(x) e^{i\omega t}$.

The real part will then provide the desired solution.

We have

$$u_t(x, t) = i\omega X(x) e^{i\omega t}$$

$$u_{xx}(x, t) = X''(x) e^{i\omega t}$$

So

$$u_t = c^2 u_{xx} \Rightarrow i\omega X(x) e^{i\omega t} = c^2 X''(x) e^{i\omega t}$$

$$\Rightarrow X''(x) = \frac{i\omega}{c^2} X(x) = \alpha^2 X(x)$$

where

$$\alpha = \pm \sqrt{\frac{i\omega}{c^2}} = \pm \sqrt{\frac{\omega}{c^2}} \frac{(1+i)}{\sqrt{2}}$$

Then

$$X(x) = A e^{\sqrt{\frac{\omega}{c^2}} (1+i)x} + B e^{-\sqrt{\frac{\omega}{c^2}} (1+i)x}$$
We want our solution to be bounded as \( x \to \infty \)

\[
A = 0
\]

\[
X(x) = B e^{-\sqrt{\frac{\omega}{2c} \cdot (1+i)x}}
\]

\[
u(x, t) = B e^{-\sqrt{\frac{\omega}{2c}} \cdot (1+i)x} e^{i \omega t}
\]

BC: \( u(0, t) = u_0 e^{i \omega t} \) \( \Rightarrow B = u_0 \)

So
\[
u(x, t) = u_0 e^{-\sqrt{\frac{\omega}{2c}} \cdot (1+i)x} e^{i \omega t}
\]

\[
u(x, t) = u_0 e^{-\sqrt{\frac{\omega}{2c}} x} e^{i (\omega t - \sqrt{\frac{\omega}{2c}} x)}
\]

We want real part:

\[
u(x, t) = u_0 e^{-\sqrt{\frac{\omega}{2c}} x} \cos(\omega t - \sqrt{\frac{\omega}{2c}} x) \quad (31.3)
\]

[Check directly that this satisfies (31.1), (31.2)
- intro. of complex res. is a device to simplify the calculation.]

(31.3) describes an **attenuated**

temperature wave - amplitude decreases exponentially as \( x \) increases.
Note also: 1) temp. at depth \( x \) is out of phase with temp. at surface \( x = 0 \).

2) the bigger is \( D \), the greater is the attenuation, and the smaller the penetration of the wave into the soil.

So:
- daily variations \( \rightarrow \) smaller penetration
- yearly variations \( \rightarrow \) larger penetration

Some numbers: \( c^2 \approx 2 \times 10^{-3} \) cm\(^2\)/sec  
  for typical soil

For yearly variations: \( D \approx \frac{2F}{(365/24)(3600)} \) sec\(^{-1}\)

At \( x \approx 1 \) m = 100 cm

\[
\sqrt{\frac{\omega}{2c^2}} \times \sim 0.7 \sim \frac{\pi}{4}
\]

\[
e^{-\sqrt{\frac{\omega}{2c^2}} x} \sim \frac{1}{2}
\]
So, at depth of 1 metre, have $\frac{\pi}{4}$ phase lag, and amplitude attenuation by $\frac{1}{2}$.

At depth of 4 metres, have $\pi$ phase lag, and attenuation by $\frac{1}{15}$.

So: in winter at 4 metres down, when is Summer at surface – and amplitude of temp. variation only a fraction of that at surface.

(⇒ usefulness of a deep cellar)

For daily variations, $D$ multiplied by factor 365. And $\sqrt{365} \approx 19$.

So damping & phase lag which for yearly variations were at $x$, now are at $\frac{x}{19}$.

So e.g. decrease of amplitude to $\frac{1}{15}$ and 'reversal of time of day' now occur at depth of about $\frac{400}{19} \approx 21$ cm.

- Daily variations only in surface skin.
These effects all in topsoil. On a different length scale altogether is effect that temperature steadily increases as bore down into Earth's crust—about 3°C per 100 m.

\[ u_x(0, t) \sim \frac{3}{100} \, \text{°C/m} = 3 \times 10^{-2} \, \text{°C/cm} \]

Lord Kelvin used this to estimate age of Earth—very controversial. He assumed Earth is a hot, chemically inert solid, cooling.

\[ u = u_0 \sim 0 \, \text{°C} \]

\( \Theta_0 \sim \) melting temp. for iron \( \sim 1200 \, \text{°C} \)

Can show effect is near surface—OK to use 'Flat Earth' approximation:

\[ u = 0 \, \text{°C} \]

\[ u(x, 0) = \Theta_0 \]
Image problem: Solved in Sec. 29 (pp. 29.9-29.12)

\[ u(x,t) = \Theta_0 \text{erf}(\frac{x}{\sqrt{4ct}}) \]

\[ \Rightarrow u_x(x,t) = \Theta_0 \left[ \frac{d}{dx} \left( \frac{x}{\sqrt{4ct}} \right) \right] \text{erf}' \left( \frac{x}{\sqrt{4ct}} \right) \]

\[ = \frac{\Theta_0}{\sqrt{4ct}} \frac{2}{\sqrt{\pi}} e^{-\frac{x^2}{4ct}} \]

\[ \Rightarrow u_x(0,t) = \frac{\Theta_0}{\sqrt{\pi c^2 t}} \]

\[ \Rightarrow \text{Time Earth has been cooling: } t = \frac{(\Theta_0)^2}{\kappa c^2 [u_x(0,t)]} \]

Everything on RHS known.
(\( c^2 \) for basalt \( \approx 6 \times 10^{-3} \) cm²/sec)

So \[ t \approx \frac{(1200)^2}{(2)(6 \times 10^{-3})(3 \times 10^{-6})^2} \text{ secs } \approx 27 \times 10^6 \text{ years} \]

Even allowing for uncertainties in data, Kelvin concluded age of Earth \( < 400 \times 10^6 \) years.
If we have a source of heat energy present, the 1-D Heat Equation is modified to

$$u_t(x,t) - c^2u_{xx}(x,t) = q(x,t) \quad \text{(31.4)}$$

\[ \text{Source term given} \]

$$p_0q(x,t)dx = \text{amount of heat energy produced in } (x, x+dx) \text{ per unit time.}$$

How can we solve (31.4), say for \(0 < x < 2\), with BCs \(u(0,t) = 0 = u(2,t), t > 0\), and with IC \(u(x,0) = f(x), 0 < x < 2\)?

One way to proceed is to expand \(u(x,t)\) and \(q(x,t)\) in Fourier series, and solve for coefficients in series for \(u\) — see example next lecture.
Summary:

1) Know how to use Method of Images for semi-infinite regions with either
   a) \( u(0, t) = 0 \) or  b) \( u_x(0, t) = 0 \)

2) Understand trick of using complex \( u(x, t) \) to solve for temperature waves in earth.

3) Follow Kelvin's estimate of Earth's age.