To deal with effects of boundaries for a finite string, we can use

**Fourier's Method: Separation of Variables and Superposition.**

Suppose string occupies $0 \leq x \leq L$

![Graph of a string with boundary conditions]

We want to solve:

\[ u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < L \quad t > 0 \]  

with

**ICs**

\[ u(x,0) = f(x), \quad 0 < x < L \]  
\[ u_t(x,0) = g(x), \quad 0 < x < L \]  

and

**BCs**

\[ u(0,t) = 0 \quad t > 0 \]  
\[ u(L,t) = 0 \quad t > 0 \]
We know general form of solution to (27.1) is \( u(x,t) = F(x+ct) + G(x-ct) \), but not easy to use that here because of BCs.

Fourier's method is very different from D'Alembert's method for infinite string.

Idea is to find all possible solutions of the homogeneous equations (27.1)+(27.3a,b) having the separated form

\[
    u(x,t) = F(x) G(t) \tag{27.4}
\]

then to use fact that we can superpose these to get a general solution of (27.1)+(27.3a,b), on which we then impose (27.2) to finish the problem.

Firstly, try (27.4) in (27.3a):

\[
    0 = u(0,t) = F(0) G(t) \]

\( \Rightarrow F(0) = 0 \)  [else \( G(t) = 0 \Rightarrow u(x,t) = 0 \) trivial sol.]

Similarly, (27.4) in (27.36) gives
\[ 0 = u(t, t) = F(t) G(t) \]
\[ \Rightarrow F(t) = 0 \]

So we have now
\[
\begin{aligned}
F(0) &= 0 \\
F(t) &= 0
\end{aligned}
\]  
\[= (27.5a) \]
\[= (27.5b) \]

Now try (27.4) in (27.1):
\[
\begin{aligned}
0 &= u(x, t) = F(x) G(t) \\
\Rightarrow u_{tt}(x, t) &= F(x) G''(t) \\
u_{xx}(x, t) &= F''(x) G(t)
\end{aligned}
\]

So
\[ u_{tt} = c^2 u_{xx} \Rightarrow \quad F(x) G''(t) = c^2 F''(x) G(t) \]

Divide b.s. by \( c^2 F(x) G(t) \):
\[ \Rightarrow \quad \frac{G''(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} \]

Now argue that \( \text{LHS cannot equal RHS} \) unless both sides equal a common constant:
\[ \frac{G''(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = k \quad (\text{const.}) \]
Thus we get two equations:
\[ \frac{G''(t)}{c^2 G(t)} = k \quad \text{and} \quad \frac{F''(x)}{F(x)} = k \]

i.e.
\[ G''(t) = k c^2 G(t) \quad \text{and} \quad F''(x) = k F(x) \]

(27.6a, b)

Consider \((27.6b) + (27.5a, b)\)

Case a) \(k = 0\)

\[
F''(x) = 0 \quad \Rightarrow \quad F(x) = ax + b
\]

Then
\[
F(0) = 0 \quad \Rightarrow \quad 0 = a \cdot 0 + b \quad \Rightarrow \quad F(x) = ax
\]
\[
F(L) = 0 \quad \Rightarrow \quad 0 = a \cdot L \quad \Rightarrow \quad a = 0 \quad \Rightarrow \quad F(x) = 0
\]

\[
\Rightarrow u(x, t) = 0 \quad - \text{trivial sol}^0.
\]

Case b) \(k > 0\), \(k = \mu^2, \text{say,}\) \((\mu > 0)\)

\[
F''(x) = \mu^2 F(x) \quad \Rightarrow \quad F(x) = a \cosh(\mu x) + b \sinh(\mu x)
\]
\[
F(0) = 0 \quad \Rightarrow \quad 0 = a + b \quad \Rightarrow \quad F(x) = b \sinh(\mu x)
\]
\[
F(L) = 0 \quad \Rightarrow \quad b \sinh(\mu L) = 0 \quad \Rightarrow \quad b = 0 \quad \Rightarrow \quad F(x) = 0
\]

\[
\Rightarrow u(x, t) = 0 \quad - \text{trivial sol}^0.
\]
Case (c) \( k < 0 \): \( k = -p^2 \), say, \( (p > 0) \)

\[
\begin{align*}
F''(x) &= -p^2 F(x) \Rightarrow F(x) = a \cos(px) + b \sin(px) \\
F(0) &= 0 \Rightarrow 0 = a + b \Rightarrow F(x) = b \sin(px) \\
F(L) &= 0 \Rightarrow 0 = b \sin(pL) \\
&\Rightarrow \text{either } b = 0 \ (\Rightarrow F(x) = 0 \Rightarrow u(x,t) = 0 \text{ trivial}) \\
&\text{or } pL = n\pi, \quad n = 1 \text{ or } 2 \text{ or } 3 \ldots \\
&\text{i.e.} \quad k = -p^2 = -\left(\frac{n\pi}{L}\right)^2 \\
&\text{and} \quad F(x) = b \sin\left(\frac{n\pi x}{L}\right) \quad n = 1 \text{ or } 2 \text{ or } 3 \ldots \tag{27.7}
\end{align*}
\]

Now to (27.6a), with this value of \( k \):

\[
G''(t) = -\left(\frac{n\pi}{L}\right)^2 G(t)
\]

\[
\Rightarrow G(t) = A \cos\left(\frac{n\pi ct}{L}\right) + B \sin\left(\frac{n\pi ct}{L}\right)
\]

Now for \( n = 1 \text{ or } 2 \text{ or } 3 \ldots \), we have a solution \( u(x,t) \) of the homogeneous equation (27.1), (27.3a, b)
\[ u(x,t) = F(x) G(t) = \left[ b \sin \left( \frac{n \pi x}{L} \right) \right] \left[ A \cos \left( \frac{n \pi c t}{L} \right) \right] \]

or

\[ u_n(x,t) = \left[ A_n \cos \left( \frac{n \pi c t}{L} \right) + B_n \sin \left( \frac{n \pi c t}{L} \right) \right] \sin \left( \frac{n \pi x}{L} \right) \]

\[ n = 1, 2, 3 \ldots \]

(27.8)

An infinite sequence of solutions!

Now invoke superposition principle:

\[ u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) \]

also satisfies (27.1) + (27.3a,b)

Fourier made bold assumption that if let \( N \to \infty \), get general soln of (27.1)+(27.3a,b)

\[ u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) \]

\[ = \sum_{n=1}^{\infty} \left[ A_n \cos \left( \frac{n \pi c t}{L} \right) + B_n \sin \left( \frac{n \pi c t}{L} \right) \right] \sin \left( \frac{n \pi x}{L} \right) \]

(27.9)
Since we want to solve \((27.1) + (27.3a, b) + (27.2a, b)\), we assume \((27.9)\) valid and impose \((27.2a, b)\) to fix the arbitrary constants \(A_n, B_n, n=1, 2,\ldots\)

\((27.2a)\):
\[
f(x) = u(x, 0) = \sum_{n=1}^{\infty} (A_n + c) \sin \left(\frac{n \pi x}{L}\right)
\]

Recognize Fourier half-range sine series for \(f(x)\)!!

\[
A_n = \frac{2}{L} \int_0^L \sin \left(\frac{n \pi x}{L}\right) f(x) \, dx
\]

\((27.10)\)

\((27.2b)\):
\[
g(x) = u_t(x, 0) = \sum_{n=1}^{\infty} \left(0 + \frac{n \pi c}{L} B_n\right) \sin \left(\frac{n \pi x}{L}\right)
\]

\[
B_n = \frac{2}{n \pi c} \int_0^L \sin \left(\frac{n \pi x}{L}\right) g(x) \, dx
\]

\((27.11)\)
We have

\[ u(x, t) = \left[ A_1 \cos\left( \frac{\pi x}{L} \right) + B_1 \sin\left( \frac{\pi x}{L} \right) \right] \sin\left( \frac{\pi t}{T} \right) \]

\[ + \left[ A_2 \cos\left( \frac{2\pi x}{L} \right) + B_2 \sin\left( \frac{2\pi x}{L} \right) \right] \sin\left( \frac{2\pi t}{T} \right) \]

\[ + \cdots \]

(27.12)

At each fixed value of \( t \), we can think of this as the Fourier half-range sine series for \( u(x, t) \). So our solution is in form of a Fourier series at each \( t \).

Another way to think of the solution is

\[ u(x, t) = \sum_{n=1}^{\infty} \left[ A_n u_n^{(A)}(x, t) + B_n u_n^{(B)}(x, t) \right] \]

\[ = A_1 u_1^{(A)}(x, t) + B_1 u_1^{(B)}(x, t) \]

\[ + A_2 u_2^{(A)}(x, t) + B_2 u_2^{(B)}(x, t) \]

\[ + \cdots \]

(27.13)
Here
\[ u^{(A)}_n(x,t) = \cos\left(\frac{n\pi ct}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \]
\[ u^{(B)}_n(x,t) = \sin\left(\frac{n\pi ct}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \]

\( n = 1, 2, 3, \ldots \)

We call these the normal modes of vibration of the string.

Note that they are periodic in time as well as space.

Recall \( \cos(\omega t) \), \( \sin(\omega t) \) have angular frequency \( \omega \); frequency (no. of cycles/unit time)
\[ \nu = \frac{\omega}{2\pi} \]
and period \( T = \frac{1}{\nu} = \frac{2\pi}{\omega} \)

So \( u^{(A)}_n(x,t) \) and \( u^{(B)}_n(x,t) \) have angular frequency \( \omega_n = \frac{n\pi c}{L} \)
frequency \( \nu_n = \frac{n\pi c}{2L} \)
period \( T_n = \frac{2L}{n\pi c} \)
We call $u_1^{(A)}$ and $u_1^{(B)}$ the **fundamental** mode of vibration (they differ by phase in time):

\[ y = \frac{c}{2L} \]

$\sin(\frac{\pi x}{L})$

\[ \gamma = \frac{c}{2L} \]

$\sin(\frac{\pi x}{L})$

$-\sin(\frac{\pi x}{L})$

\[ v_2 = 2v_1 = \frac{c}{L} \]

$u_2^{(A)}$ and $u_2^{(B)}$ constitute the **first overtone**:  

and so on. For example, to get 'middle C', adjust tension $T$, string density $p$ and length $L$  

So $v_1 = \frac{c}{2L} = \frac{c}{2L} \sqrt{\frac{T}{\rho}} = 278.4375 \text{ cycles/sec} (= \text{Hertz})$
Then first overtone has frequency
\[ \nu_2 = 2 \nu_1 = 556.875 \text{ Hz} \]
This is C above middle C.
So (27.13) expresses the solution as an infinite series in the normal modes.
It represents the harmonic analysis of the string displacement as a function of \( x \) and \( t \).
In practice the coefficients \( A_n \) and \( B_n \) may decrease in size very quickly with increasing \( n \), so a good approximation to the solution is given by the first few terms of (27.12) or (27.13), just as the first few terms of the half-range series for \( f \) and \( g \) on p. (27.7) may give good approximations to those given functions which determine the solution.
Summary:

1) Work very carefully through Fourier's Method (= Separation of Variables) - must be able to go from (27.1, 23) through to (27.9, 19, 11) in an exam.

2) Understand meaning of solution, as Fourier half-range sine series at each value of time t, and as a series in terms of normal modes.