The 3 PDEs

$$\frac{\partial u(x,t)}{\partial t} - c^2 \frac{\partial^2 u(x,t)}{\partial x^2} = 0 \quad (26.1)$$

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0 \quad (26.2)$$

$$\frac{\partial^2 u(x,t)}{\partial t^2} - c^2 \frac{\partial^2 u(x,t)}{\partial x^2} = 0 \quad (26.3)$$

are all linear and homogeneous $\Rightarrow$ the superposition principle applies in each case:

If $u = u_1$ and $u = u_2$ satisfy (26.1), so does

$$u = \alpha u_1 + \beta u_2$$

$\alpha, \beta$ arbitrary constants.

Similarly for (26.2) and also for (26.3).
PDEs have a much richer set of solutions than ODEs:

Easy to check that, for example,

\[ u_1 = e^{-\alpha^2 t} \sin(\alpha x), \quad u_2 = x^3 + 6c^2 x t \]

are each solutions of (26.1);

\[ u_1 = \cos(\alpha x) \sinh(\alpha y), \quad u_2 = e^{-\alpha x} \sin(\alpha y), \quad u_3 = \alpha (x^3 - 3xy^2) \]

are each solutions of (26.2); and

\[ u_1 = x^2 + c^2 t^2, \quad u_2 = e^{\alpha (x - ct)} \]

\[ u_3 = \sinh(\alpha x) \cos(\alpha t) \]

are each solutions of (26.3). It is not so easy to find general solutions. But we can do it for the 1-D wave equation, (26.3).
\[
\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}
\]

We change the independent variables:

Let \( \nu = x + ct \quad \zeta = x - ct \)

\( \leftrightarrow \quad x = \frac{1}{2} (\nu + \zeta), \quad t = \frac{1}{2c} (\nu - \zeta) \)

Then

\[
\frac{\partial}{\partial x} = \frac{\partial \nu}{\partial x} \frac{\partial}{\partial \nu} + \frac{\partial \zeta}{\partial x} \frac{\partial}{\partial \zeta} = \frac{\partial}{\partial \nu} + \frac{\partial}{\partial \zeta}
\]

\[
\frac{\partial}{\partial t} = \frac{\partial \nu}{\partial t} \frac{\partial}{\partial \nu} + \frac{\partial \zeta}{\partial t} \frac{\partial}{\partial \zeta} = c \frac{\partial}{\partial \nu} - c \frac{\partial}{\partial \zeta}
\]

Then

\[
\frac{\partial^2 u^2}{\partial t^2} = (c \frac{\partial}{\partial \nu} - c \frac{\partial}{\partial \zeta})(c \frac{\partial}{\partial \nu} - c \frac{\partial}{\partial \zeta}) u
\]

\[
= c^2 \left( \frac{\partial^2 u}{\partial \nu^2} - 2 \frac{\partial^2 u}{\partial \zeta \partial \nu} + \frac{\partial^2 u}{\partial \zeta^2} \right)
\]

\[
\frac{\partial^2 u}{\partial x^2} = (\frac{\partial}{\partial \nu} + \frac{\partial}{\partial \zeta})(\frac{\partial}{\partial \nu} + \frac{\partial}{\partial \zeta}) u
\]

\[
= \left( \frac{\partial^2 u}{\partial \nu^2} + 2 \frac{\partial^2 u}{\partial \zeta \partial \nu} + \frac{\partial^2 u}{\partial \zeta^2} \right)
\]
So \[ \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \]

\[ \Rightarrow c^2 \left( \frac{\partial^2 u}{\partial t^2} - 2 \frac{\partial^2 u}{\partial t \partial x} + \frac{\partial^2 u}{\partial x^2} \right) - c^2 \left( \frac{\partial^2 u}{\partial t^2} + 2 \frac{\partial^2 u}{\partial t \partial x} + \frac{\partial^2 u}{\partial x^2} \right) = 0 \]

\[ \Rightarrow -4c^2 \frac{\partial^2 u}{\partial t \partial x} = 0 \]

\[ \frac{\partial^2 u}{\partial t \partial x} = 0 \quad (26.5) \]

(26.5) is equivalent to (26.3) — but can see how to solve (26.5)!

\[ \frac{\partial}{\partial x} \left[ \frac{\partial u}{\partial t} \right] = 0 \]

\[ \Rightarrow \frac{\partial u}{\partial t} = A(t) \quad A \text{ arbitrary fn.} \]

\[ u = \int A(v) \, dv + G(x) \quad G \text{ arbitrary fn.} \]

\[ u = F(v) + G(x) \]

\[ \Rightarrow u(x, t) = F(x + ct) + G(x - ct) \quad F, G \text{ arbitrary fns. (twice differentiable)} \]
General solution of 1-D wave equation:

\[ u(x,t) = F(x+ct) + G(x-ct) \] (26.6)

F, G arbitrary functions of a single variable.

\textbf{Meaning:} F(x+ct) represents a wave of constant shape travelling to \underline{left} at speed c:

At \( t > 0 \):

\[ u = F(x) \]

At \( t = 0 \):

\[ u = F(x+ct) \]

Similarly, \( G(x-ct) \) represents a wave of constant shape travelling to \underline{right} at speed c.

We call c \underline{the wave speed}. \[ \text{[Recall } c = \sqrt{\frac{T}{\rho}} \text{]} \]

To fix a particular solution of the wave equation, we have to give enough info. to fix the two functions F and G.
Initial Value Problem (IVP) for infinite string.

Suppose string is very long, and we are looking near middle—ends so far away we can ignore end effects (= boundary conditions). In effect, we consider
\[
\frac{\partial^2 u(x,t)}{\partial t^2} - c^2 \frac{\partial^2 u(x,t)}{\partial x^2} = 0, \quad -\infty < x < \infty, \quad t > 0
\]

Aside on notation: Let's agree to write
\[
\frac{\partial u(x,t)}{\partial t} = u_t(x,t), \quad \frac{\partial u(x,t)}{\partial x} = u_x(x,t),
\]
\[
\frac{\partial^2 u(x,t)}{\partial t \partial x} = u_{tx}(x,t), \quad \frac{\partial^2 u(x,t)}{\partial x^2} = u_{xx}(x,t)
\]
and so on.

So our PDE is \[u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0.\]

Now suppose we have ICs:
\[
\begin{align*}
  u(x,0) &= f(x) \quad \text{given fns.,} \quad -\infty < x < \infty, \\
  u_t(x,0) &= g(x) \quad \text{f: initial shape,} \quad g: \text{initial velocity profile}
\end{align*}
\]
We know solution of PDE + ICs must be of form
\[ u(x,t) = F(x+ct) + G(x-ct) \] (26.7)
for some F and G. Can we determine F and G from given f and g?

Well, (26.7) \( \Rightarrow \) \[ u_t(x,t) = \frac{\partial F(x+ct)}{\partial t} + \frac{\partial G(x-ct)}{\partial t} \]
\[ = \frac{\partial (x+ct)}{\partial t} F'(x+ct) + \frac{\partial (x-ct)}{\partial t} G'(x-ct) \]
i.e.
\[ u_t(x,t) = c F'(x+ct) - c G'(x-ct) \] (26.8)

Now, using ICs at \( t=0 \):
(26.7) \( \Rightarrow \) \[ f(x) = u(x,0) = F(x) + G(x) \] (26.9)
(26.8) \( \Rightarrow \) \[ g(x) = u_t(x,0) = c F'(x) - c G'(x) \] (26.10)

Integrating (26.10) w.r.t x we get:
\[ c F(x) - c G(x) = \int g(x) dx \]
or better,
\[ F(x) - G(x) = \frac{1}{c} \int_{x_0}^{x} g(s) ds + A \] (26.11)
Now can use (26.9) and (26.11) to fix $F$ and $G$:

Adding: \[ 2F(x) = f(x) + \frac{1}{c} \int_{x_0}^{x} g(s) \, ds + A \]

Subtracting: \[ 2G(x) = f(x) - \frac{1}{c} \int_{x_0}^{x} g(s) \, ds - A \]

\[ F(x+ct) + G(x-ct) = \frac{1}{2} f(x+ct) + \frac{1}{2c} \int_{x_0}^{x+ct} g(s) \, ds + \frac{1}{2} A \]
\[ + \frac{1}{2} f(x-ct) - \frac{1}{2c} \int_{x_0}^{x-ct} g(s) \, ds - \frac{1}{2} A \]

Now \[ \int_{a}^{b} - \int_{a}^{c} = \int_{c}^{b}, \quad \infty \]

\[ u(x,t) = F(x+ct) + G(x-ct) \]

\[ u(x,t) = \frac{1}{2} f(x+ct) + \frac{1}{2} f(x-ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) \, ds \]  

D'Alembert's solution for

IVP A 1-D wave equation on whole $x$-axis.
EX: I Cs \( u(x,0) = f(x) \), \( u_t(x,0) = 0 = g(x) \)

(string released from rest with initial shape given by \( f(x) \))

\[ u(x,t) = \frac{1}{2} f(x+ct) + \frac{1}{2} f(x-ct) \]

Get pulses to left and right, each at speed \( c \), each with shape \( \frac{1}{2} f \):

Eg.

\[ \begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet} \\
\end{array} \]

\[ \begin{array}{c}
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\end{array} \]

'Plucked string'
EX: 2  ICs: \[ u(x, 0) = 0 = f(x) \]
\[ u_x(x, 0) = Ae^{-x^2} = g(x) \]

D'Alambert's solution:

\[ u(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} Ae^{-s^2} \, ds \]

- can't evaluate

\[ = \frac{A \sqrt{\pi}}{4c} \left\{ \frac{2}{\sqrt{\pi}} \int_{0}^{x+ct} e^{-s^2} \, ds - \frac{2}{\sqrt{\pi}} \int_{0}^{x-ct} e^{-s^2} \, ds \right\} \]

Introduce new function:

\[ \text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_{0}^{z} e^{-s^2} \, ds \]
- the error function
Then our solution is

\[ u(x,t) = \frac{A}{4\sqrt{\pi c}} \left( \text{erf}(x+ct) - \text{erf}(x-ct) \right) \]

How does it look? How does \( \text{erf} \) look?

Some properties of \( \text{erf}(z) \):

1) \( \text{erf}(-z) = -\text{erf}(z) \quad \text{odd function} \)

\[ \text{Pf:} \quad \text{erf}(-z) = -\frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-s^2} \, ds \]

\[ (\text{Put } u = -s, \quad dv = -ds) \]

\[ s = -z \leftrightarrow v = z \]
\[ s = 0 \leftrightarrow v = 0 \]

\[ = -\frac{2}{\sqrt{\pi}} \int_{0}^{z} e^{-v^2} (-dv) \]

\[ = -\text{erf}(z). \]

2) \( \frac{d}{dz} \text{erf}(z) (\equiv \text{erf}'(z)) = \frac{2}{\sqrt{\pi}} e^{-z^2} > 0 \)

\( \Rightarrow \text{erf}(z) \) is monotonically increasing
3) \( \text{erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-s^2} ds = \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-s^2} ds = 1 \)

(Then \( \Rightarrow \text{erf}(-\infty) = -1 \))

4) \( \text{erf}(0) = 0 \)

5) \( \text{erf}(\frac{1}{2}) \approx 0.52 \), \( \text{erf}(1) \approx 0.84 \),
\( \text{erf}(2) \approx 0.995 \), \( \text{erf}(3) \approx 0.99998 \)

So, our solution to 1-D Wave Equation:
Summary:

1) PDEs have wide variety of solutions
2) Know (26.1, 2, 3). They are linear, homogeneous PDEs – superposition principle applies!
3) Know derivation of general solution of 1-D wave equation and D'Alembert's solution of IVP
4) Understand last two examples.

K § 11.9, 11.13, 11.2
K § 12.1, 12.2, 12.4