A conservation law:

Consider some substance flowing through space, with density \( p(x,y,z,t) \) and flux vector field \( \mathbf{j}(x,y,z,t) \):

\[
\text{Amount of substance in } \delta V \text{ at } (x,y,z) \text{ at time } t = p(x,y,z,t) \delta V
\]

\[
\text{Amount of substance flowing per unit time over } \delta S \text{ at } (x,y,z) = \mathbf{j}(x,y,z,t) \cdot \mathbf{n} \delta S
\]
If substance is conserved (not being created or destroyed), then for every closed surface $S$, and corresponding enclosed 3-D region $V$, we must have

$$\frac{d}{dt} \iiint_{V} \rho \, dv = - \iint_{S} \mathbf{J} \cdot n \, ds$$

(amount in $V$ at time $t$)

(rate at which amount in $V$ is increasing)

(flux into $V$ over $S$ per unit time)

Using Gauss's Theorem

$$\iiint_{V} \frac{\partial \rho}{\partial t} \, dv = - \iint_{S} (\nabla \cdot \mathbf{J}) \, dv$$
\[ \Rightarrow \iiint (\frac{\partial P}{\partial t} + \nabla \cdot \mathbf{J}) \, dv = 0 \]

Since \( S \) and corresponding \( P \) are arbitrary, it follows that at every point \( (x, y, z) \) we have

\[ \frac{\partial P(x, y, z, t)}{\partial t} + \nabla \cdot \mathbf{J}(x, y, z, t) = 0 \quad (25.1) \]

Conservation of 'substance'

**EX:** 1) **Moving fluid, velocity field** \( \mathbf{v}(x, y, z, t) \)

If fluid is incompressible, mass density \( \rho(x, y, z, t) = \rho_0 \) (const.)

Since mass flux vector field is \( \rho \mathbf{v} \), we have

\[ \frac{\partial \rho_0}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}) = 0 \Rightarrow \rho_0 \nabla \cdot \mathbf{v} = 0 \]

\[ \Rightarrow \nabla \cdot \mathbf{v} = 0 \]

Conservation of mass of incompressible fluid.
Important example for rest of course: -

Conduction of heat in a uniform solid.

\( u(x,y,z,t) = \) temperature at \( (x,y,z) \) at time \( t \)

\( \rho = \) mass/unit volume of solid
\( \sigma = \) specific heat of solid \( (\text{amount of heat energy to raise unit mass through one unit of temperature}) \)

\[ \Rightarrow \rho \sigma u(x,y,z,t) = \text{heat energy density} \]

Heat energy flux vector
\[ J(x,y,z,t) = -\chi \nabla u(x,y,z,t) \]
thermal conductivity

[Heat flows from hotter to cooler regions, "down the temperature gradient"]
Conservation of heat energy:

\[
\frac{\partial}{\partial t} [p \rho u(x, y, z, t)] + \nabla \cdot \left[ -k \nabla u(x, y, z, t) \right] = 0
\]

\[
\frac{\partial u(x, y, z, t)}{\partial t} - \frac{k}{c^2} \nabla \cdot \left[ \nabla u(x, y, z, t) \right] = 0
\]

\[k = c^2\]

\[
\frac{\partial u(x, y, z, t)}{\partial t} = c^2 \nabla \cdot \left[ \nabla u(x, y, z, t) \right]
\]  
(25.2)

The heat (conduction) equation.

\[c^2 : \text{thermal diffusivity}\]

Consider \(\nabla \cdot \left[ \nabla u \right] = \nabla \cdot \left( \frac{\partial u}{\partial x} i + \frac{\partial u}{\partial y} j + \frac{\partial u}{\partial z} k \right)\)

\[\left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \cdot \left( \frac{\partial u}{\partial x} i + \frac{\partial u}{\partial y} j + \frac{\partial u}{\partial z} k \right)\]

\[= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\]

Write this as

\[\nabla^2 u \quad (= \text{del}^2 u)\]
\[ \nabla^2 = \nabla \cdot \nabla = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \]

\[ = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]

\[ \nabla \] vector differential operator
\[ \nabla^2 = \nabla \cdot \nabla \] scalar differential operator

\[ \nabla^2 \text{ called "the Laplacian"} \]

So the heat equation (25.2) says

\[
\frac{\partial u(x, y, z, t)}{\partial t} = c^2 \left[ \frac{\partial^2 u(x, y, z, t)}{\partial x^2} + \frac{\partial^2 u(x, y, z, t)}{\partial y^2} + \frac{\partial^2 u(x, y, z, t)}{\partial z^2} \right]
\]

\[ = c^2 \nabla^2 u(x, y, z, t) \quad (25.3) \]

A partial differential equation (PDE)

4 independent variables \( x, y, z, t \)
1 dependent variable \( u \)
1st-order in \( t \) \quad 2nd-order in \( x, y \) and \( z \)
Sometimes call (25.3) the 3-D heat equation.
If no y or z dependence in problem, get

$$\frac{\partial u(x,t)}{\partial t} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \quad (25.4)$$

1-D heat equation.

**Ex:**

- **u(x,t)** insulated sides
- **u=x=0**
- **u=u_1**
- **u=u_2**

Iron bar, initially at constant temperature. Then one face held at temp. **u=u_1**, other at **u=u_2**.

**u(x,t)** satisfies (25.4) for

$$t > 0 \text{ and } 0 < x < L.$$  

**Typical boundary/initial value problem:**

Find solution with IC **u(x,0) = u_0**

BCs **u(0,t) = u_1**, **u(L,t) = u_2**.
The heat equation governs the way temperature varies as heat flows from hotter to cooler regions, in a way consistent with conservation of heat energy.

Consider a 3-D region, with a prescribed, constant (in time) temperature distribution on the surface:

\[ u(x,y,z) \text{ prescribed on } \Gamma \]

After a long time, we expect temperature inside to approach a steady distribution \( u(x,y,z) \) satisfying (from (25.3))

\[ \frac{\partial u(x,y,z)}{\partial t} = c^2 \nabla^2 u(x,y,z) \]
Laplace's Equation

- describes equilibrium (steady-state) temperature distributions.

If no z-dependence in problem,

\[ \frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = 0 \quad (25.6) \]

2-D Laplace Equation

PDE with 2 indep. variables \( x, y \) and one dependent variable \( u \).

One more PDE with 2 indep. variables and one dep. variable for us to consider.

Consider small displacements of stretched string, whose equilibrium position is along \( x \)-axis.
Consider small segment:

\[ u(x,t) \]

\[ u(x+\delta x,t) \]

Assume tension \( T \) in string has constant magnitude

\[
\text{Force on segment in } +u\text{-direction} \\
= -T \sin \theta_1 - T \sin \theta_2 \\
= -T \theta_1 - T \tan \theta_2 \\
= \frac{\partial u(x,t)}{\partial x} + T \frac{\partial u(x+\delta x,t)}{\partial x} \\
\approx -T \left[ \frac{\partial^2 u(x,t)}{\partial x^2} + \delta x \frac{d^2 u(x,t)}{\partial x^2} \right] \\
= T \frac{d^2 u(x,t)}{\partial x^2} \delta x
\]
If mass/unit length of string is $g$, then mass x accel. of segment in $u$-direction is

$$ (P \delta x) \frac{\partial^2 u(x,t)}{\partial t^2} $$

Then, by Newton's 2nd Law,

$$ T \frac{\partial^2 u(x,t)}{\partial x^2} \delta x = (P \delta x) \frac{\partial^2 u(x,t)}{\partial t^2} $$

$$ \Rightarrow \quad \frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \quad (25.7) $$

$$ c^2 = \frac{T}{g} $$

1-D wave equation

[The 3-D wave equation is

$$ \frac{\partial^2 u(x,y,z,t)}{\partial t^2} = c^2 \nabla^2 u(x,y,z,t) ]$$

We will consider the 3 PDE's in one dependent, and two independent variables.
The 1-D heat equation:
\[ \frac{\partial u(x,t)}{\partial t} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \]

Laplace's Equation in 2-D:
\[ \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0 \]

The 1-D wave equation:
\[ \frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \]

Summary:

1) Understand idea of a conservation law and know (25.1)
2) Know definition of Laplacian operator \( \nabla^2 \)
3) Know forms of heat eqns, wave eqns and Laplace's eqns in 1, 2, 3 dimensions

K. pp. 511, 512, 5111, 1112 K. pp. 407, 408, 411
464-5 
§12.1, 12.2