Consider a periodic function $f(x)$ defined for all $x$, with period $p > 0$.

$$f(x + p) = f(x), \quad -\infty < x < \infty$$

Fourier's Idea: Analyse in terms of the elementary functions with period $p$:

$$1, \cos\left(\frac{2\pi x}{p}\right), \cos\left(\frac{4\pi x}{p}\right), \ldots$$

$$\sin\left(\frac{2\pi x}{p}\right), \sin\left(\frac{4\pi x}{p}\right), \ldots$$

The birth of Harmonic Analysis!

Contrast with idea of synthesis.
So Fourier wrote:

\[ f(x) = a_0 + a_1 \cos \left( \frac{2 \pi x}{p} \right) + b_1 \sin \left( \frac{2 \pi x}{p} \right) \]
\[ + a_2 \cos \left( \frac{4 \pi x}{p} \right) + b_2 \sin \left( \frac{4 \pi x}{p} \right) \]
\[ + \ldots \ldots \]

and found how to calculate the coefficients \( a_0, a_1, a_2, \ldots, b_1, b_2, \ldots \)
to make it work.

The RHS of (19.1) is called the Fourier Series (expansion) of \( f(x) \),
and \( a_0, a_1, a_2, \ldots, b_1, b_2, \ldots \)
are called the corresponding Fourier coefficients.
EX:

\[ f(x) = 2x, \quad 0 < x < 1 \]
\[ f(x+1) = f(x), \quad -\infty < x < \infty \]

This function is defined for all \(-\infty < x < \infty\), and is periodic with period 1. It is piecewise continuous.

In this case, the relevant elementary functions have period 1: They are

\[ 1, \cos(2\pi x), \cos(4\pi x), \ldots \text{ and } \sin(2\pi x), \sin(4\pi x), \ldots \]
The Fourier analysis of \( f(x) \) is:

\[
f(x) = 1 + \frac{-2}{\pi} \sin(2\pi x) + \frac{-1}{\pi} \sin(4\pi x) + \frac{-2}{3\pi} \sin(6\pi x) + \frac{-1}{2\pi} \sin(8\pi x) + \cdots
\]
Successive approximations:

\[ f(x) \]

\[ 1 - \frac{2}{3} \sin(2\pi x) \]

\[ 1 - \frac{2}{3} \sin(2\pi x) - \frac{1}{15} \sin(4\pi x) \]

\[ 1 - \frac{2}{3} \sin(2\pi x) - \frac{1}{15} \sin(4\pi x) - \frac{1}{30} \sin(6\pi x) \]
In general, given \( f(x) \), \(-\infty < x < \infty\), with period \( p \)

(i.e. \( f(x+p) = f(x) \), \(-\infty < x < \infty\)), we try to expand as

\[
f(x) = a_0 + a_1 \cos \left( \frac{2\pi x}{p} \right) + a_2 \cos \left( \frac{4\pi x}{p} \right) + \ldots
\]
\[+ b_1 \sin \left( \frac{2\pi x}{p} \right) + b_2 \sin \left( \frac{4\pi x}{p} \right) + \ldots
\]

OK

Setting \( p = 2L \) for convenience,

\[
f(x) = a_0 + a_1 \cos \left( \frac{\pi x}{L} \right) + a_2 \cos \left( \frac{2\pi x}{L} \right) + \ldots
\]
\[+ b_1 \sin \left( \frac{\pi x}{L} \right) + b_2 \sin \left( \frac{2\pi x}{L} \right) + \ldots
\]

OK

\[
f(x) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right) \right]
\]

The Fourier analysis of \( f(x) \)

The Fourier (series) expansion of \( f(x) \)
The first and most important question facing us is:

How do we find the Fourier coefficients \( a_0, a_1, a_2 \ldots \), \( b_1, b_2 \ldots \) when we are given \( f(x) \)?

[The second and more difficult question is: When does it all work? That is, when can we find \( a_0, a_1, a_2 \ldots \), \( b_1, b_2 \ldots \) such that the RHS of (19.2) equals (converges to) the LHS at each value of \( x \)? We postpone discussion of this second question until later.]

To answer the first question, we make use of orthogonality relations amongst the elementary periodic functions:—
\[ \int_{-\pi}^{\pi} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) \, dx \quad (m \neq n) \]

Use \( \cos A \cos B = \frac{1}{2} \left[ \cos(A+B) + \cos(A-B) \right] \)
So we have \( \frac{1}{L} \int_{-L}^{L} \left\{ \cos \left[ \frac{(m+n)\pi x}{L} \right] + \cos \left[ \frac{(m-n)\pi x}{L} \right] \right\} \, dx \)

\[ = \frac{1}{L} \left\{ \frac{L}{(m+n)\pi} \sin \left[ \frac{(m+n)\pi x}{L} \right] \bigg|_{x=-L}^{x=L} + \frac{L}{(m-n)\pi} \sin \left[ \frac{(m-n)\pi x}{L} \right] \bigg|_{x=-L}^{x=L} \right\} \]

\[ = 0 \]

Consider \( \int_{-L}^{L} \sin \left( \frac{m\pi x}{L} \right) \sin \left( \frac{n\pi x}{L} \right) \, dx \quad (m \neq n) \)

Use \( \sin A \sin B = \frac{1}{2} \left[ \cos (A-B) - \cos (A+B) \right] \)
to get

\[ \frac{1}{L} \int_{-L}^{L} \left\{ \cos \left[ \frac{(m-n)\pi x}{L} \right] - \cos \left[ \frac{(m+n)\pi x}{L} \right] \right\} \, dx \]
\[
\frac{L}{(m-n)\pi} \left\{ \begin{array}{l}
\sin[(m+n)\pi x] \\
\frac{L}{(m+n)\pi} \sin[(m-n)\pi x]
\end{array} \right\}^{x=L}_{x=-L} = 0
\]

Consider
\[
\int_{-L}^{L} \sin \frac{m\pi x}{L} \cos \frac{n\pi x}{L} \, dx \quad (m \neq n)
\]

Use
\[
\sin A \cos B = \frac{1}{2} \left[ \sin (A+B) + \sin (A-B) \right]
\]

to get
\[
\frac{1}{2} \int_{-L}^{L} \left\{ \sin \frac{(m+n)\pi x}{L} + \sin \frac{(m-n)\pi x}{L} \right\} \, dx
\]

\[
= -\frac{1}{2} \left\{ \frac{L}{(m+n)\pi} \left[ \cos \frac{(m+n)\pi x}{L} \right]^{x=L}_{x=-L} + \frac{L}{(m-n)\pi} \left[ \cos \frac{(m-n)\pi x}{L} \right]^{x=L}_{x=-L} \right\}
\]

\[
= 0 + 0 \quad (\text{as} \ \cos \theta = \cos (-\theta))
\]

\[
= 0
\]
Finally, consider
\[ \int_{-L}^{L} \sin \left( \frac{m\pi x}{L} \right) \cos \left( \frac{n\pi x}{L} \right) \, dx \]

\[ = \frac{1}{2} \int_{-L}^{L} \sin \left( \frac{(2m\pi)x}{L} \right) \, dx \]

\[ = -\frac{1}{2} \left( \frac{L}{2m\pi} \right) \left[ \cos \left( \frac{2m\pi x}{L} \right) \right]_{x=-L}^{x=L} \]

\[ = 0 \]

---

We say that each one of the functions
\[ 1, \cos \left( \frac{\pi x}{L} \right), \cos \left( \frac{2\pi x}{L} \right), \ldots, \sin \left( \frac{\pi x}{L} \right), \sin \left( \frac{2\pi x}{L} \right), \ldots \]

is orthogonal to every other one.

(Or they are mutually orthogonal)

\[ \int_{-L}^{L} f_1(x) f_2(x) \, dx = 0 \]
\[ \text{Compare with orthogonal vectors: } \]
\[ i = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ \vdots \end{array} \right), \quad j = \left( \begin{array}{c} 0 \\ 1 \\ 0 \\ \vdots \end{array} \right), \quad k = \left( \begin{array}{c} 0 \\ 0 \\ 1 \\ \vdots \end{array} \right), \quad \ldots \]
\[ i \cdot j = 0, \quad i \cdot k = 0, \quad j \cdot k = 0 \quad \text{etc.} \]

Now compare (19.2) with writing an arbitrary vector \( \underline{u} \) as
\[ \underline{u} = c_1 \underline{i} + c_2 \underline{j} + c_3 \underline{k} + \ldots \]

**Summary:**

1) Idea of analysing a function with period \( p \) into a series of elementary functions with period \( p \).

2) Orthogonality of elementary periodic functions with a given period \( p = 2L \).

K 11.11.2 pp. 477 - 483, 487 - 489
K 10.1 pp. 530, 531, 534