Chaos

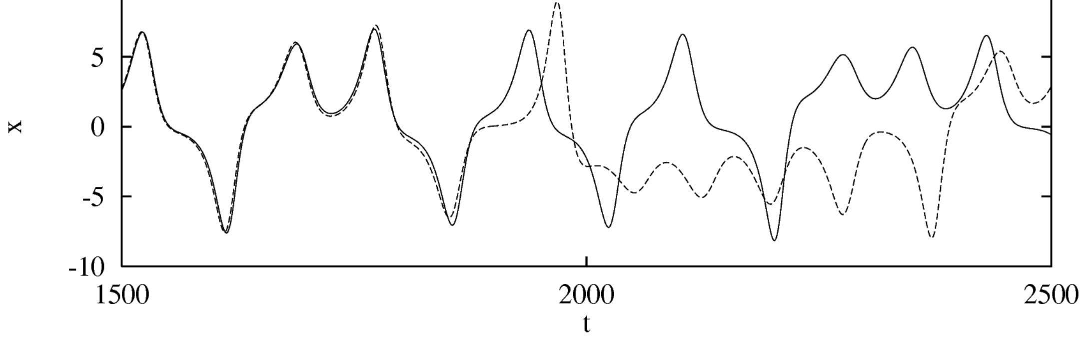
Chaos is fundamental to understanding complex systems. It lies at the border between mathematical analysis and the computational techniques used in complex systems research.

In this lecture we will look at chaotic systems including demonstrations of the Lorenz system, the double pendulum and, in more detail, the simple but powerful example of the logistic map.

1.1 Lorenz

In 1962, Edward Lorenz was studying a simplified weather model. Lorenz's model had three variables. He could plug arbitrary values into the equations, iterate them on a computer, and then stop the simulation.

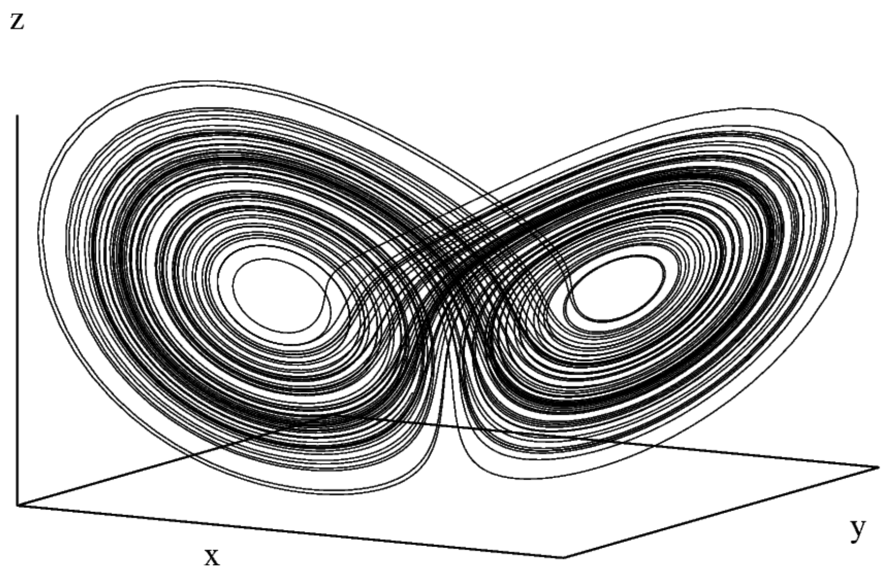
While running a particularly interesting simulation, Lorenz had to stop the system prematurely. Later, he wanted to continue the simulation where he left off so he entered in the last known values and continued the simulation. The simulation continued as it had before, but with a slight deviation. Small differences soon became large differences, and in a short time the system was nowhere near where it had been before.



In 1962, everyone believed that all dynamical systems ultimately settled into predictable patterns. How had Lorenz's small system produced such different behaviour on the second run?

What happened was that the printout truncated the last few digits of the variables so that when Lorenz re-entered the values there was a tiny difference that eventually grew to create totally different system behaviour. The system exhibited sensitive dependence on initial conditions. Lorenz had discovered chaos!

Lorenz wrote up his experience for a journal article, but it took a decade or so for his discovery to become known in the physics and mathematics community.



A trajectory in phase space for the Lorenz system:

Try predicting the behaviour of this!

1.2 Defining Chaos

- **Chaos** is the irregular, unpredictable behaviour of deterministic, nonlinear dynamical systems.
    - In a **deterministic system** the time evolution is exactly specified. There is no external randomness.
    - **Nonlinear systems** contain processes whose behaviour is not directly proportional to the input. Instead there may be a quadratic or more complicated relationship.
- Chaos is a nonlinear phenomenon that appears in a broad range of real world and mathematical models.
- Computational techniques are essential to investigating chaos since most nonlinear systems are impossible to solve analytically.
- Chaos is characterised by sensitive dependence on initial conditions which results in unpredictable behaviour.

  The tiniest effect can grow until the system's future becomes completely unpredictable. This is known as the butterfly effect.

  "If a butterfly flaps its wings in Tokyo, then a month later it may cause a hurricane in Brazil."
- Chaos is everywhere in the real world. Applications include cellular biology, economics, weather, quantum physics…

1.3.1 Linear Maps

- A *map* is just a function such that for every element of one set there is a unique element of another set.
- Before we get to the logistic map, here are some simple, linear maps. Can you predict their behaviour at some distant time?
    - $x(t+1) = 2\,x(t)$
    - $x(t+1) = -x(t)$
    - $x(t+1) = x(t)/2 + 1/2$
    - $x(t+1) = 1 - x(t)/2$

  - Does x oscillate, grow unbounded, approach a fixed point or behave chaotically?
  - Think of a system that the behaviour of each equation could represent.

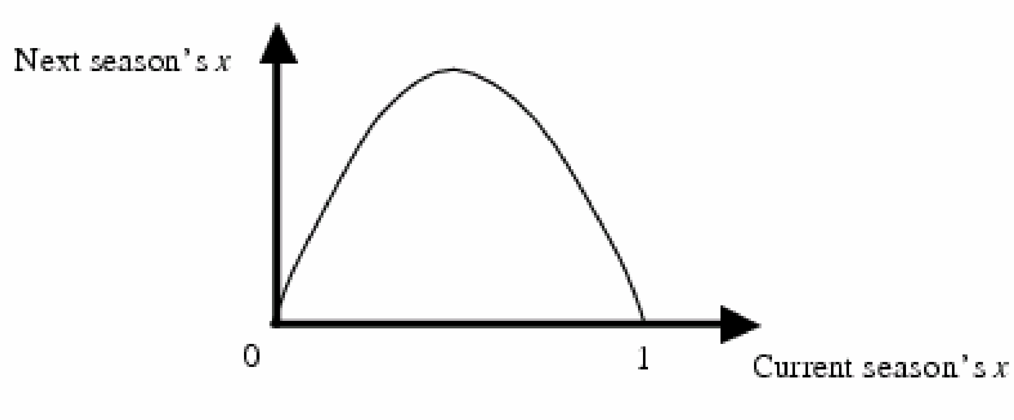
1.3.2 The Logistic Map

- This simple nonlinear system exhibits rich chaotic behaviour and allows us to get an intuitive feel for how chaos works.
- The logistic map is defined by the iterative equation:

$$x(t+1) = r\, x(t)\, (1-x(t))$$

- It models population growth in a limited environment:
  - x is the population size divided by the carrying capacity of the environment
  - r is the reproduction rate (r is between 0 and 4)
- The model assumes that the reproduction rate is dependent on the available resources. When the population is high the available resources are reduced so the reproductive success is reduced.

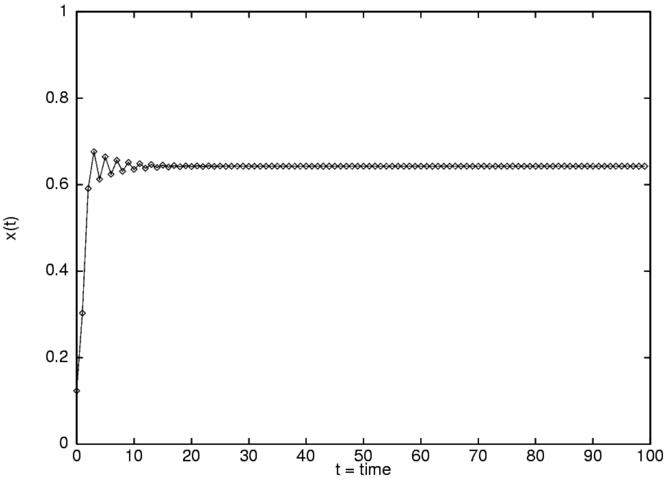
  - What happens at $t+1$ if $x(t)=0$?
  - What happens at $t+1$ if $x(t)=1$?



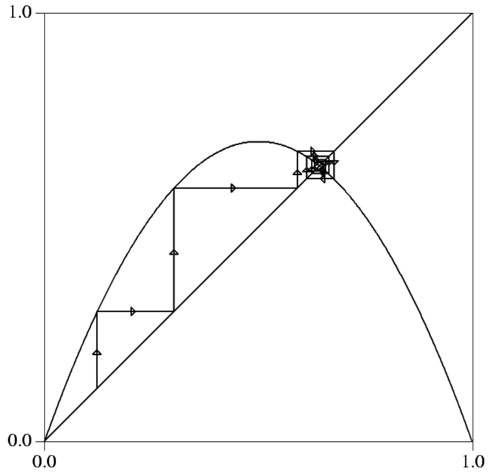
The curve $x(t+1) = r\, x(t)\, (1-x(t))$.

1.3.3 Dynamics

- Phase space trajectories represent a 'trace' of the dynamical behaviour of the system (like a long exposure photograph).
- Phase space easily identifies stable fixed points or periodic dynamics.
- For 1D systems the phase space is (*x(t),x(t+1)*).
- Dynamics in phase space: go to the curve (find *x(t+1)* from *x(t)*), then go to diagonal (update *x(t)*). Repeat.
- Fixed points exist where the function intersects the line *x(t+1)=x(t)*.



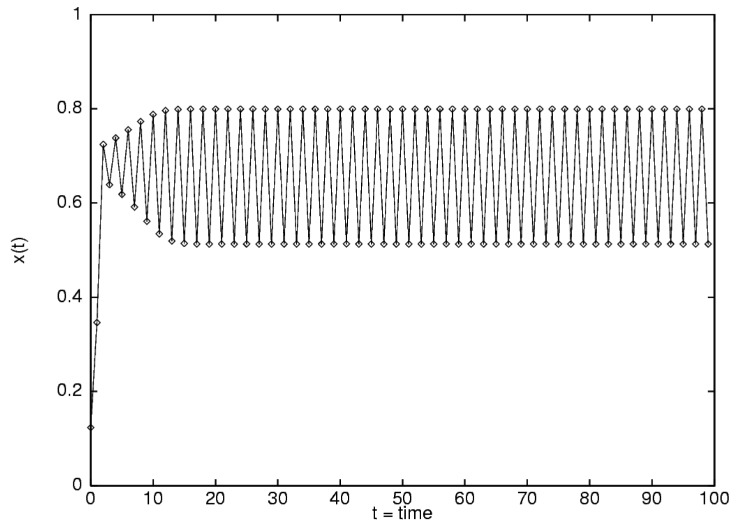
(a)



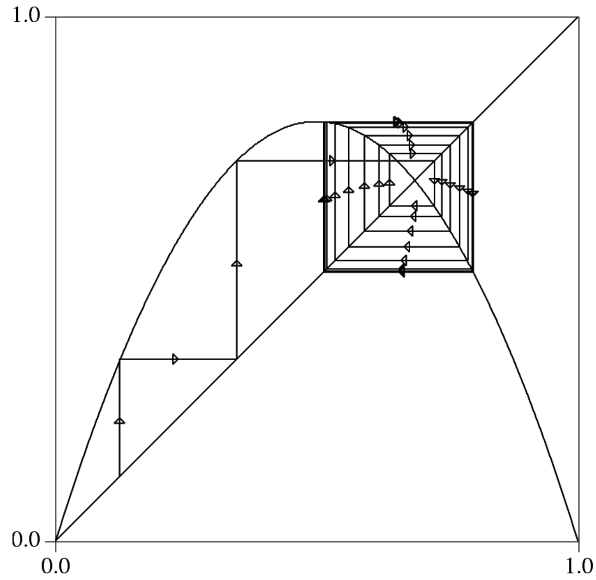
(b)

Fixed point dynamics, *r=2.8*

Fixed point is stable (see phase space (b)).



(a)



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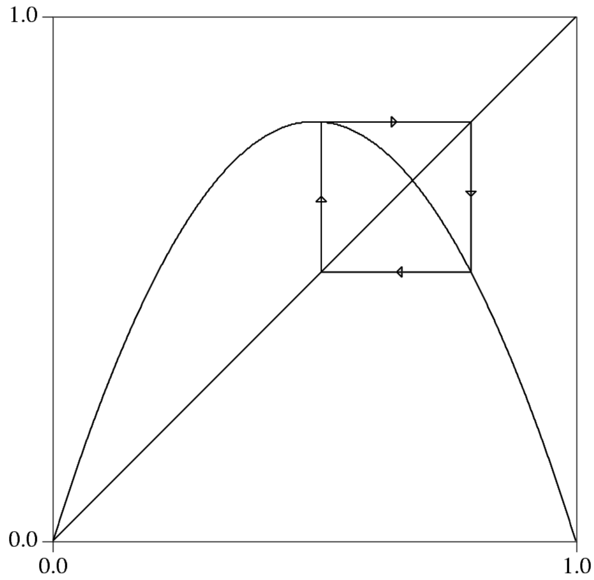
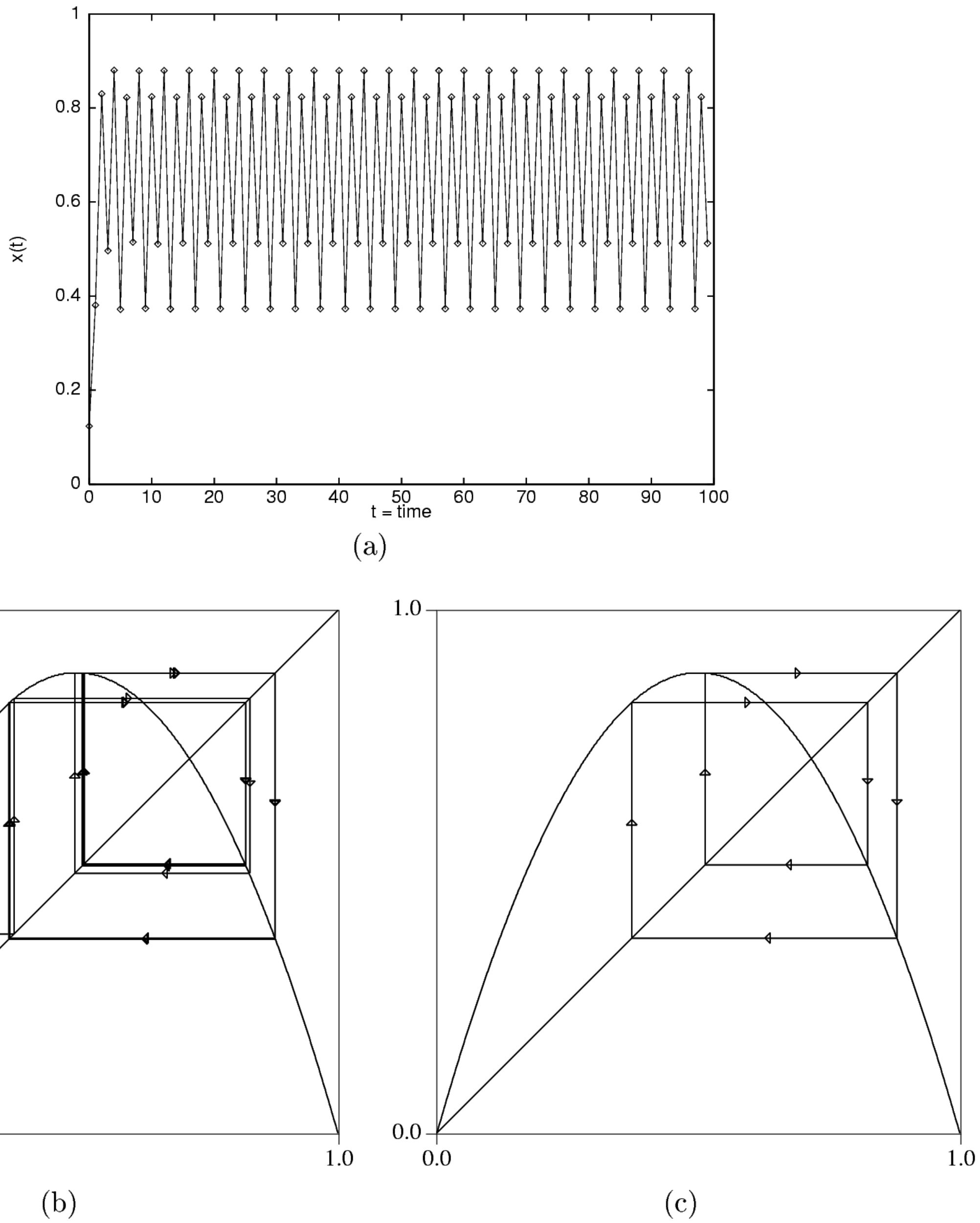


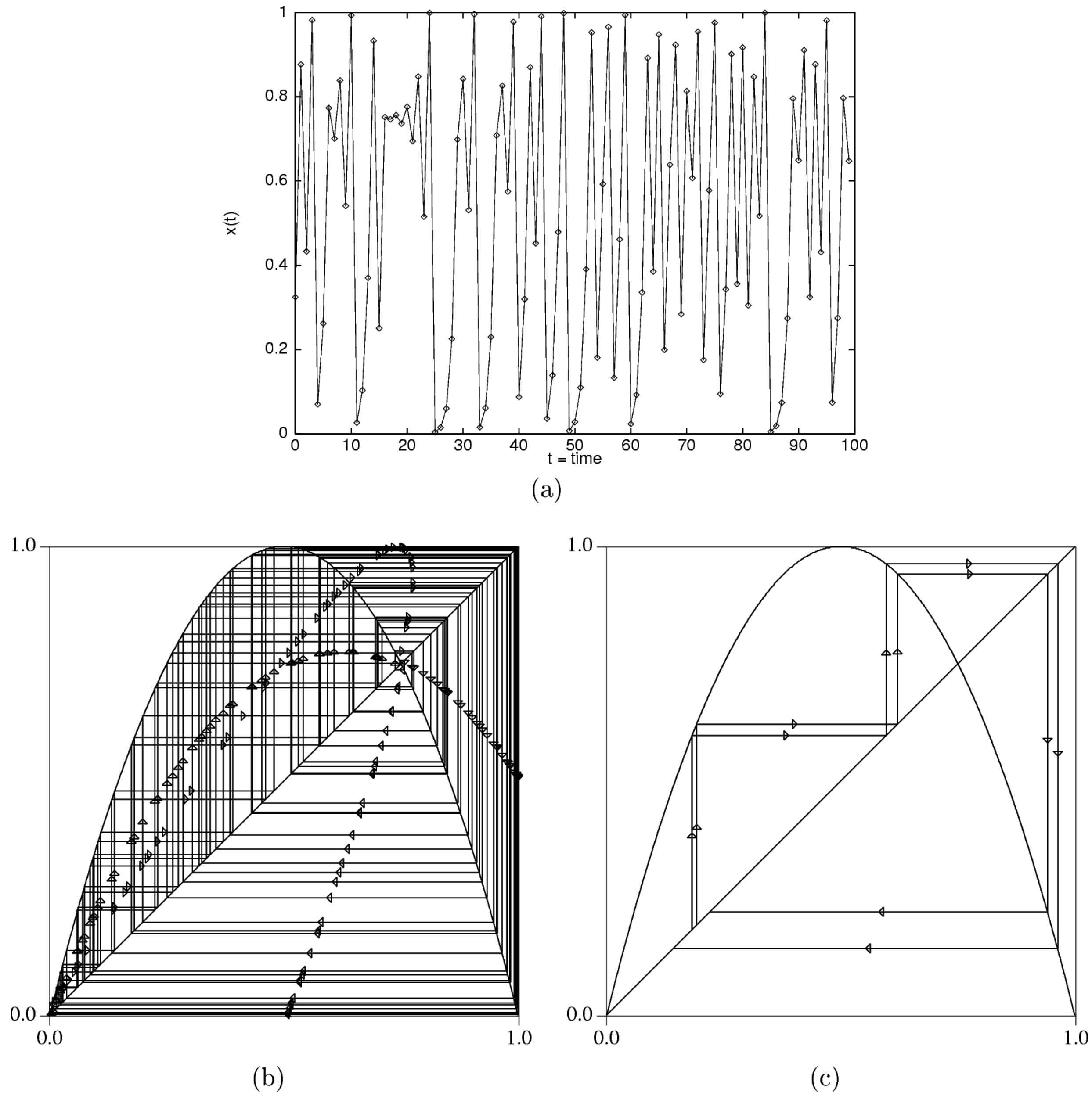
(c)

Period-two dynamics, *r=3.2*

Fixed point is unstable (see (b)) but period-two cycle is stable.



Period-four dynamics, *r=3.52*



Chaotic dynamics, *r=4*

Phase space (b). Nearby trajectories diverge (c).

- The system doesn't grow unbounded, approach a fixed point or become periodic. It is chaotic.

o Can you still make short term predictions?

1.3.4 Fixed Points

- For all discrete systems, fixed points can be found by setting:

$$x(t) = x(t+1) = x^*, \text{ say,}$$

and solving for x^*.

- For the logistic map:

$$x(t+1) = f(x(t)) = r\,x(t)(1-x(t)),$$

so solve:

$$x^* = r\,x^*(1-x^*).$$

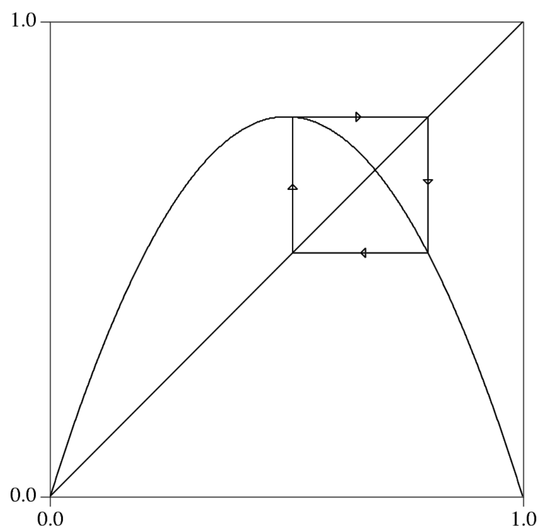
- o How do you find the period-two orbit?

1.3.5 Stability

- If the magnitude of the gradient at the fixed point, $|f'(x^*)|$, is less than 1, the fixed point is stable and attracting.

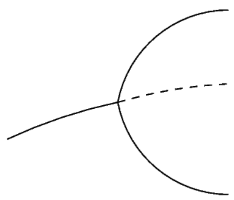
  If it is greater than 1, it is unstable and repelling.

- This holds for all discrete systems.

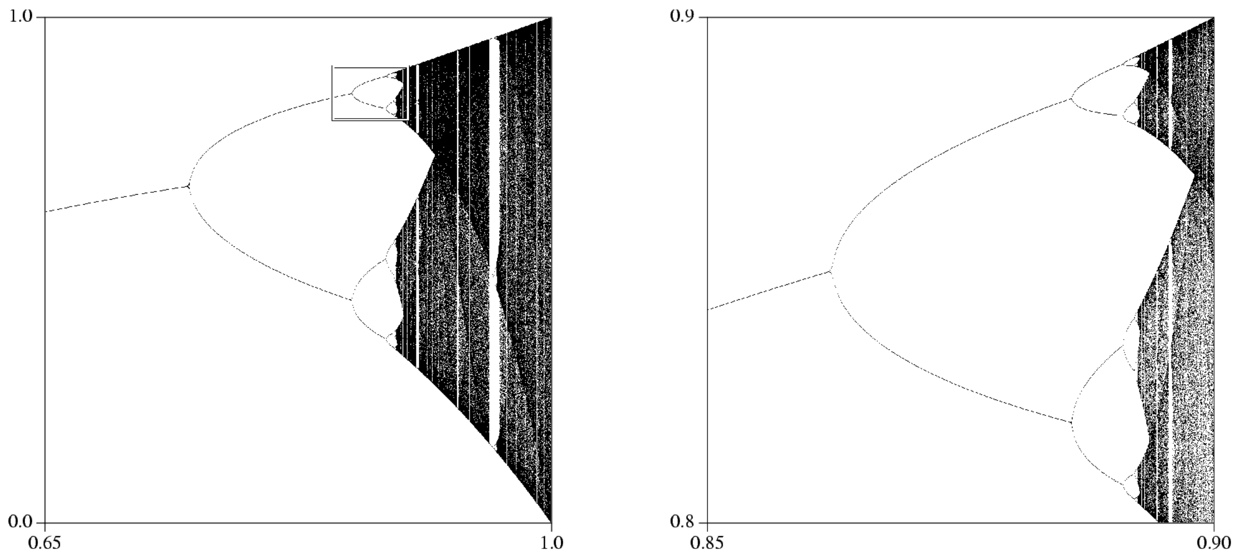


1.3.6 Bifurcation

- As r increases the value of the fixed point increases. At some critical r the fixed point becomes unstable and the system displays period-two behaviour with the system oscillating between two values (solid lines). The old fixed point still exists but it is now unstable (dotted line). As r increases further the period-two cycle will bifurcate again producing period-four behaviour.



- Each period doubling spans a shorter amount of space than the previous doubling. Eventually the period doublings converge to what looks like an infinite-period attractor.



- Notice the self-similarity of the bifurcation diagram.

1.4 Complex Systems

- Simple systems don't necessarily display simple behaviour.
- We have seen unpredictable chaotic motion in mathematical systems with only a few interacting variables.
- What happens in systems with hundreds of interacting parts?
- What type of behaviour do complex systems display?