1. (a) (i) The relation is antisymmetric. For all \( a, b \in \{0, 1, 2, 3\} \), if \((a, b) \in \rho \) and \( a \neq b \) then \((b, a) \notin \rho \).

(ii) The relation is not antisymmetric. \((2, 1) \in \rho \) and \((1, 2) \in \rho \) but \( 1 \neq 2 \).

(b) There are three distinct equivalence classes: \( [a]_\rho = \{a\} \), \([b]_\rho = \{b, d\} \), and \([c]_\rho = \{c\} \).

2. (a) The relation \( R_3 \) is reflexive: \( \forall x \in \mathbb{Z}, m \mid (x^n - x^m) \) since \( m \mid 0 \) for all \( m \in \mathbb{Z} \) with \( m \geq 1 \). The relation is symmetric: Suppose that \( a S b \) for \( a, b \in \mathbb{Z} \). Then \( m \mid (a^n - b^m) \), so \((a^n - b^m) = mk \) for some integer \( k \). But then \( b^n - a^n = -(a^n - b^m) = -(mk) = (-m)k \), and \(-m \in \mathbb{Z} \) since \( m \in \mathbb{Z} \). Thus \( m \mid (b^n - a^n) \) and \( b Sa \).

The relation is transitive: Suppose that \( a S b \) and \( b Sc \) for \( a, b, c \in \mathbb{Z} \). Then \( m \mid (a^n - b^m) \), so \((a^n - b^m) = mk \) for some integer \( k \), and \( m \mid (b^m - c^n) \), so \((b^m - c^n) = ml \) for some integer \( l \). But then \( a^n - c^n = (a^n - b^m) + (b^m - c^n) = mk + ml = m(k + l) \), and \( k + l \in \mathbb{Z} \) since \( k, l \in \mathbb{Z} \). Thus \( m \mid (a^n - c^n) \) and \( a Sc \). Thus the relation is reflexive, symmetric and transitive, so it is an equivalence relation.

(b) There are two distinct equivalence classes: \( [0]_{S^*} = \{x \in \mathbb{Z} \mid x = 3k \) for some integer \( k \} \) and \([1]_{S^*} = [2]_{S^*} = \{x \in \mathbb{Z} \mid x = 3k + 1 \) or \( x = 3k + 2 \) for some integer \( k \} \).

3. (a) \( S \) is reflexive: \( \forall a \in \mathbb{Z}, a + 1 > a \).

\( S \) is antisymmetric: Suppose that \( a S b \) and \( b Sa \) for \( a, b \in \mathbb{Z} \). Therefore \( a + 1 > b \) and \( b + 1 > a \), and hence \( b + 1 > a > b - 1 \) from which it is clear that \( a = b \).

\( S \) is transitive: Suppose that \( a S b \) and \( b Sc \) for \( a, b, c \in \mathbb{Z} \). Therefore \( a + 1 > b \) and \( b + 1 > c \). So \( a + 1 > b \geq c \) and hence \( a Sc \).

Since \( S \) is reflexive, antisymmetric and transitive, \( S \) is a partial order relation.

(b) \( S \) is total. Either \( x \geq y \), in which case \( x + 1 > y \) and \( x Sy \), or \( x < y \), in which case \( y + 1 > x \) and \( y S x \).

4. (a) (i) \( R_1 \) is reflexive. Every prime factor of \( a \) is a prime factor of \( a \), for each \( a \in \mathbb{Z}^+ \).

(ii) \( R_1 \) is not symmetric. Every prime factor of \( 6 \) is a prime factor of \( 30 \), but not every prime factor of \( 30 \) is a prime factor of \( 6 \).

(iii) \( R_1 \) is not antisymmetric. Every prime factor of \( 3 \) is also a prime factor of \( 9 \), but every prime factor of \( 9 \) is also a prime factor of \( 3 \) (and \( 3 \neq 9 \)).

(iv) \( R_1 \) is transitive. For all \( a, b, c \in \mathbb{Z}^+ \), if every prime factor of \( a \) is a prime factor of \( b \) and every prime factor of \( b \) is a prime factor of \( c \) then every prime factor of \( a \) is a prime factor of \( c \).

(v) \( R_1 \) is not symmetric, therefore it is not an equivalence relation.

(vi) \( R_1 \) is not antisymmetric, therefore it is not a partial order.

(b) (i) \( R_2 \) is reflexive. For all \( a \in \mathbb{Z}^+ \), \( \frac{a}{a} = 1 \in \mathbb{Z} \).

(ii) \( R_2 \) is not symmetric. \((6, 2) \in R_2 \) but \((2, 6) \notin R_2 \).

(iii) \( R_2 \) is antisymmetric. If \((a, b), (b, a) \in R_2 \) then \( a = bk \) for some \( k \in \mathbb{Z}^+ \) and \( b = al \) for some \( l \in \mathbb{Z}^+ \). So \( a = alk \) and hence \( 1 = lk \) and \( l = k = 1 \). Hence \( a = b \).

(iv) \( R_2 \) is transitive. If \((a, b), (b, c) \in R_2 \) then \( a = bk \) for some \( k \in \mathbb{Z}^+ \) and \( b = cl \) for some \( l \in \mathbb{Z}^+ \). So \( a = clk \) and \( lk \in \mathbb{Z} \). Hence \((a, c) \in R_2 \).

(v) \( R_2 \) is not symmetric, therefore not an equivalence relation.

(vi) \( R_2 \) is reflexive, antisymmetric and transitive, therefore it is a partial order.

5. Here are the arrow diagrams for (a) and (b).

(a) Range of \( f = \{x, y, z, w\} \). The function is one-to-one and onto.
Range of \( f = \{x, y\} \). \( f \) is not one-to-one since \( f(a) = f(b) \). \( f \) is not onto since there is no \( p \in \{a, b, c, d\} \) such that \( f(p) = z \).

6. (a) \( f \) is not one-to-one. \( f(\{0\}) = f(\{1\}) \) but \( \{0\} \neq \{1\} \).
(b) \( f \) is not onto. There is no element \( x \in \mathcal{P}(A) \) such that \( f(x) = |X| = 6 \).
(c) Range of \( f = \{0, 1, 2, 3\} \).

(the end)