8 Sequences and Series

- A sequence is an ordered set of numbers, and we call the numbers in the sequence the terms of the sequence.

- Some sequences have an identifiable pattern, allowing us to develop a formula to calculate any particular term of the sequence directly, or a formula to calculate the value of a term from the values of preceding terms.

- As well as illustrating the beauty of mathematics, sequences have many applications; for instance, the calculation of mortgage repayments and estimating population growth.

- A series is the sum of the terms of a sequence. We will investigate mathematical induction, a proof technique that is useful in proving results about series.

- Topics in this section are:
  - Introduction to sequences
  - Arithmetic and geometric sequences
  - Arithmetic and geometric series
  - Applications to population growth
  - Mathematical induction.
• It is important to be aware that specifying the first few terms of a sequence does not adequately describe the sequence.

• You probably thought that the next term of the sequence in the previous example was 32. The sequence with closed form

\[ a_n = 2^n \quad (n = 1, 2, \ldots) \]

certainly has 2, 4, 8, 16, 32 as its first five terms.

• However, in the previous example, if we had chosen the closed form

\[ a_n = (n - 1)(n - 2)(n - 3)(n - 4) + 2^n \quad (n = 1, 2, \ldots), \]

then the first five terms of the sequence would be

2, 4, 8, 16, 56.

• Simply writing down the first few terms of a sequence is never enough to specify the remainder of the sequence!

• In order to describe a sequence, if you cannot write down all the terms of the sequence, then you must give a general description of the sequence such as a closed form or a recurrence relation with initial conditions.
• A general rule that allows you to calculate the terms of a sequence can be specified in two ways.

<table>
<thead>
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<th>n</th>
<th>0</th>
<th>1</th>
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<th>3</th>
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<tbody>
<tr>
<td>a_n</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>32</td>
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• A formula that allows direct calculation of the value of any particular term of a sequence is called a closed form for the sequence. For example, the sequence 3, 5, 7, 9, 11 has closed form \(3 + 2n\) \((n = 0, 1, 2, 3, 4)\).

\(n = 1, 2, 3, 4, 5 \rightarrow 2n + 1\)

• A formula that specifies how to calculate the value of a term using the value(s) of one (or more) preceding terms is called a recurrence relation. A recursive definition of a sequence is a recurrence relation combined with initial conditions giving the value(s) of one (or more) initial terms of the sequence. For example, the sequence 3, 5, 7, 9, 11 can be defined recursively as \(a_n = a_{n-1} + 2\) \((n = 1, 2, 3, 4)\) with initial condition \(a_0 = 3\).

**Example 8.1.2** Find both a closed form and a recursive definition of a sequence whose first four terms are 2, 4, 8, 16.

**Closed form:**

\[a_1 = 2 \quad a_0 = 2\]
\[a_2 = 4 \quad a_1 = 4\]
\[a_3 = 8 \quad a_2 = 8\]
\[a_4 = 16 \quad a_3 = 16\]

\[a_n = 2^n \quad a_n = 2^{n+1}\]

**Recursive definition:**

\[a_n = 2a_{n-1}, \quad [a_1 = 2]\]
\[a_2 = 2a_{2-1} = 2a_1 = 2 \cdot 2 = 4\]
\[a_3 = 2a_{3-1} = 2a_2 = 8\]
\[;\]
8.1 Introduction to sequences

- A sequence is an ordered set of numbers.
- We call each number in a sequence a term of the sequence.
- A sequence with a fixed number of terms is a finite sequence, and a sequence with an infinite number of terms is an infinite sequence.
- To describe a sequence, we usually choose a particular letter and combine that letter with subscripts to represent the terms of the sequence. For example, in the sequence

\[ a_1, a_2, a_3, \ldots, a_{10} \]

\(a_1\) represents the first term, \(a_2\) the second term, and so on, and \(a_{10}\) is the 10\(^{th}\) and final term of the sequence. The notation \(a_n\) represents a general term of the sequence, and we use the notation \(\{a_n\}\) or \(\{a_n\}_{n=1}^{10}\) to represent the entire sequence.
- The symbol \(\infty\) is used for infinity, and so an infinite sequence might be written as \(\{a_n\}_{n=1}^{\infty}\).

\[ \text{Example 8.1.1} \quad \text{Let } \{a_n\}_{n=0}^{\infty} \text{ be a sequence with } a_n = 2n - 1. \]
List the first four terms of the sequence.

\[ a_0 = 2 \times 0 - 1 = -1 \]
\[ a_1 = 2 \times 1 - 1 = 1 \]
\[ a_2 = 2 \times 2 - 1 = 3 \]
\[ a_3 = 2 \times 3 - 1 = 5 \]

(We could also write this sequence as \(\{2n - 1\}_{n=0}^{\infty}\).)
8.2 Arithmetic and geometric sequences

- An *arithmetic sequence* (A.S.) is a sequence in which the difference between any two successive terms is a constant.
- The difference between successive terms in an arithmetic sequence is called the *common difference*.
- For example, the A.S. with first term 2 and common difference 3 starts with the five terms
  \[ 2, \ 5, \ \ 8, \ 11, \ 14. \]
  The A.S. with first term 10 and common difference \(-5\) starts with the five terms
  \[ 10, \ 5, \ 0, \ -5, \ -10. \]

**Example 8.2.1** Determine whether each of the following could be the first four terms of an arithmetic sequence.

a) \[ 1, \ 2 \frac{1}{2}, \ 4, \ 5 \frac{1}{4} \]
   \[ \text{\L_2 \ L_1 \ L_3 \ L_4} \]
   \[ \text{X} \]

b) \[ 5, \ 2, \ -1, \ -4 \]
   \[ \text{\text{L_1 \ L_2 \ L_3 \ L_4}} \]
   \[ \text{\checkmark} \]

c) \[ 3, \ 4 + 2x, \ 5 + 4x, \ 6 + 6x \]
   \[ \text{\L_3 \ L_2 \ L_1 \ L_2} \]
   \[ \text{\checkmark} \]
• If we know the values of the first term and the common difference of an arithmetic sequence, we can determine a recursive definition or a closed form for the sequence.

Let \( \{a_n\}_{n=1}^{\infty} \) be an arithmetic sequence with first term \( a \) and common difference \( d \). Then

\[
a_{n+1} = a_n + d \quad (n = 1, 2, \ldots) \quad \text{and} \quad a_1 = a
\]

is a recursive definition of the A.S.

• Let \( \{a_n\}_{n=1}^{\infty} \) be an arithmetic sequence with first term \( a \) and common difference \( d \). Then

\[
a_1 = a \\
a_2 = a + d \\
a_3 = a + d + d \\
a_4 = a + d + d + d \\
a_5 = a + d + d + d + d
\]

Thus,

\[
a_n = a + (n-1)d, \quad (n = 1, 2, \ldots),
\]

is a closed form for the A.S.

Example 8.2.2 The first three terms of an A.S. are \( 5, -2, -9 \).

Determine the 100th term of the sequence.

\[
a_n = 5 \\
d = -7
\]

\[
a_{100} = 5 + 99(-7) = 5 - 693 = -688
\]

\[
a_n = 5 + (-7)n + 7 = 12 - 7n
\]
Example 8.2.3 The third term of an A.S. is 7 and the twelfth term is 10. Determine a closed form for the sequence.

\[ a_3 = 7 \quad \quad d = \frac{10 - 7}{12 - 3} = \frac{3}{9} = \frac{1}{3} \]
\[ a_{12} = 10 \]

\[ a_n = a + (n - 1)d = 7 + (n - 1) \cdot \frac{1}{3} \]
\[ a_1 = a_3 - 2d = 7 - 2 \cdot \frac{1}{3} = \frac{20}{3} \]

A geometric sequence (G.S.) is a sequence in which the ratio between any two successive terms is a constant, that is, every term is equal to the previous term multiplied by a particular number.

The ratio between successive terms in a geometric sequence is called the common ratio. \( r \).

For example, the G.S. with first term 2 and common ratio 3 starts with the five terms

\[ 2, 6, 18, 54, 162, \]

since \( \frac{6}{2} = \frac{18}{6} = \frac{54}{18} = \frac{162}{54} = 3 \).

The G.S. with first term 20 and common ratio \( -\frac{1}{2} \) starts with the five terms

\[ 20, -10, 5, -2 \frac{1}{2}, 1 \frac{1}{4} \].
Example 8.2.4 Determine whether each of the following could be the first four terms of a geometric sequence.

a) 27, 9, 3, -1.

\[
\frac{9}{27} = \frac{1}{3}, \quad \frac{3}{9} = \frac{1}{3}, \quad \frac{-1}{3} \neq \frac{1}{3}
\]

\[\times\]

b) 2, x, \(\frac{1}{2}x^2\), \(\frac{1}{4}x^3\).

\[
\frac{x}{2} = \frac{1}{2} \frac{x}{2} = \frac{1}{2^2} x = \frac{1}{2} x, \quad \frac{\frac{1}{2} x^2}{x} = \frac{1}{2} x
\]

\(\checkmark\)

\[\frac{\frac{1}{4} x^3}{\frac{1}{2} x^2} = \frac{1}{2} \frac{2}{4} = \frac{1}{4} \frac{2}{2} = \frac{1}{4} \times \frac{2}{2} = \frac{1}{2} \frac{2}{2} = \frac{1}{2}
\]

- If we know the values of the first term and the common ratio of a geometric sequence, we can determine a recursive definition or a closed form for the sequence.

- Let \(\{a_n\}_{n=1}^{\infty}\) be a geometric sequence with first term \(a\) and common ratio \(r\). Then

\[
a_{n+1} = a_n \times r \quad (n = 1, 2, \ldots) \quad \text{and} \quad a_1 = a
\]

is a recursive definition of the G.S.

- Let \(\{a_n\}_{n=1}^{\infty}\) be a geometric sequence with first term \(a\) and common ratio \(r\). Then

\[
\begin{align*}
a_1 &= a \\
a_2 &= a \times r \\
a_3 &= a \times r \times r \\
a_4 &= a \times r \times r \times r
\end{align*}
\]

so \(a_2 = ar \), so \(a_3 = ar^2 \), so \(a_4 = ar^3 \)

Thus,

\[
a_n = ar^{n-1} \quad (n = 1, 2, \ldots)
\]

is a closed form for the G.S.
Example 8.2.5  The first three terms of a G.S. are $-\frac{1}{3}, 1, -3$.
Determine the 10th term of the sequence.

\[
\begin{align*}
1 &= \frac{1}{3} \cdot -\frac{1}{3} = -3 \cdot \frac{1}{3} \\
\Rightarrow -1 &= \frac{1}{3} \\
&= -3.
\end{align*}
\]

\[
a_1 = -\frac{1}{3}, \quad r = -1.
\]

\[
a_n = a_1 \cdot r^{n-1} = -\frac{1}{3} \cdot (-3)^{n-1}.
\]

\[
a_{10} = -\frac{1}{3} \cdot (-3)^9
\]

\[
= 6561.
\]

Example 8.2.6  The first three terms of a G.S are 4, 8, 16.
Which term would be the first to exceed one million?

\[
a_1 = 4, \quad r = 2.
\]

\[
\begin{align*}
1,000,000 &< 4 \cdot 2^{n-1} \\
2^{n-1} &> 250,000
\end{align*}
\]

\[
\ln 2^{n-1} > \ln 250,000 \quad \Rightarrow \quad n-1 > \frac{\ln 250,000}{\ln 2}
\]

\[
n > 17.93 \ldots + 1
\]

\[
n > 18.93 \ldots
\]

\[
\therefore \quad \text{The 19th term will be the first one to exceed 1,000,000.}
\]
8.3 Arithmetic and geometric series

- A series is the sum of the terms of a sequence. If \( \{a_n\}_{n=1}^{N} \) is a sequence, then \( \sum_{n=1}^{N} a_n \) is the corresponding series.

- We often use the notation \( S_n \) to denote the sum of the first \( n \) terms of a sequence. If \( n \) is less than the total number of terms in the sequence, then \( S_n \) is a called a partial sum of the sequence.

- We can graph the terms of a sequence \( \{a_n\} \) with \( n \) on the horizontal axis and \( a_n \) on the vertical axis. If we draw a rectangle of width 1 and height \( a_n \) for each term \( a_n \), then the partial sum corresponds to the area given by the sum of the areas of the rectangles (where a rectangle below the horizontal axis has negative area).

\[
\begin{align*}
(a) & \quad a_n = 2n + 1 \\
& \quad (n = 1, 2, \ldots) \\
& \quad S_6 = 48.
\end{align*}
\]

\[
\begin{align*}
(b) & \quad a_n = 8 - 2n \\
& \quad (n = 1, 2, \ldots) \\
& \quad S_6 = 6.
\end{align*}
\]

\[
\begin{align*}
(c) & \quad a_n = 3^n \\
& \quad (n = 1, 2, \ldots) \\
& \quad S_5 = 363.
\end{align*}
\]

\[
\begin{align*}
(d) & \quad a_n = \left(\frac{1}{2}\right)^n \\
& \quad (n = 1, 2, \ldots) \\
& \quad S_5 = 0.96875.
\end{align*}
\]
Arithmetic series

- If \( \{a_j\}_{j=1}^{\infty} \) is an A.S. with first term \( a \) and common difference \( d \), then

\[
S_n = a_1 + a_2 + \cdots + a_n = \sum_{j=1}^{n} a_j = \frac{n}{2} (2a + (n - 1)d).
\]

**Proof** Let \( S_n \) be the sum of the first \( n \) terms of the A.S. Thus

\[
S_n = a + (a + d) + \cdots + (a + (n - 2)d) + (a + (n - 1)d).
\]

Reversing the order of the series (which won’t change the sum), we get

\[
S_n = (a + (n - 1)d) + (a + (n - 2)d) + \cdots + (a + d) + a.
\]

Adding these two equations together, we see that

\[
2S_n = (a + a + (n - 1)d) + (a + d + a + (n - 2)d) + \cdots + (a + (n - 2)d + a + d) + (a + (n - 1)d + a)
\]

\[
\Rightarrow \quad 2S_n = (2a + (n - 1)d) \cdot n + (2a + (n - 1)d) + \cdots + (2a + (n - 1)d) + (2a + (n - 1)d)
\]

\[
\Rightarrow \quad S_n = \frac{n}{2} (2a + (n - 1)d).
\]

Therefore,

\[
S_n = \frac{n}{2} (2a + (n - 1)d).
\]
Example 8.3.1 Calculate the sum of the first $n$ natural numbers.

First natural no. is 1. \( a = 1 \),
\[ d = 1 \]

\[
S_n = \frac{n}{2} \left( 2a + (n-1)d \right)
= \frac{15}{2} \left( 2 + n - 1 \right)
= \frac{15}{2} \left( n + 1 \right)
= \frac{n(n+1)}{2}
\]

\[ S_3 = \frac{3(3+1)}{2} = 6 \]

Example 8.3.2 An arithmetic sequence has first term 3 and common difference \( 2 \frac{1}{3} \). If the corresponding arithmetic series is equal to 3250, calculate the number of terms in the sequence.

\[ a = 3, \quad S_n = 3250 \]
\[ d = 2 \frac{1}{3} \]

\[
S_n = \frac{n}{2} \left( 2a + (n-1)d \right) = \frac{n}{2} \left( 2 \times 3 + (n-1) \times 2 \frac{1}{3} \right).
\]

\[
3250 = \frac{n}{2} \left( 6 + 2 \frac{1}{3} n - 2 \frac{1}{3} \right).
\]

\[
3250 = \frac{n}{2} \left( 2 \frac{1}{3} n + 3 \frac{2}{3} \right).
\]

\[
6500 = \frac{1}{3} \left( 2 \frac{1}{3} n + 3 \frac{2}{3} \right).
\]

\[
6500 = 2 \frac{1}{3} n^2 + 3 \frac{2}{3} n.
\]

\[
6500 = \frac{7}{3} n^2 + \frac{11}{3} n.
\]
\[0 = 7n^2 + 11n - 19500\]

\[n = \frac{-11 \pm \sqrt{11^2 - 4 \times 7 \times -19500}}{2 \times 7}\]

\[= \frac{-11 \pm \sqrt{121 + 546000}}{14}\]

\[= \frac{-11 \pm 739}{14}\]

\[= \frac{728}{14} \text{ or } \frac{-750}{14}\]

\[= 52 \text{ or } -52.5\]

\[\text{There are 52 terms in the sequence.}\]
Geometric series

- If \( \{a_j\}_{j=1}^{\infty} \) is a geometric sequence with first term \( a \) and common ratio \( r \), then

\[
S_n = a_1 + a_2 + \cdots + a_n = \sum_{j=1}^{n} a_j = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}.
\]

**Proof** Let \( S_n \) be the sum of the first \( n \) terms of the G.S. Thus

1. \( S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} \).

Multiplying both sides of this equation by \( r \), we get

2. \( rS_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n \).

Subtracting the first equation from this one, we see that

\[
rS_n - S_n = (ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n) - (a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1})
\]
\[
= ar^n - a.
\]

Therefore, \( (r - 1)S_n = a(r^n - 1) \), so

\[
S_n = \frac{a(r^n - 1)}{r - 1}.
\]

By multiplying the numerator and denominator of this formula by \(-1\), we obtain the alternative formula

\[
S_n = \frac{a(1 - r^n)}{1 - r}.
\]
Example 8.3.3  A geometric sequence has first three terms 100, 94, 88.36. Calculate the fewest number of terms needed for the corresponding series to exceed 1000.

\[
\begin{align*}
\alpha &= 100 \\
\Gamma &= \frac{94}{100} = \frac{88.36}{94} = 0.94, \\
S_n &= \frac{\alpha (1 - r^n)}{1 - r} \\
S_{14} &= \frac{100 (1 - 0.94^{14})}{1 - 0.94} \\
&= \frac{965.79}{0.06} \\
&\approx 16095.17
\end{align*}
\]

\[
\begin{align*}
1000 &< \frac{100 (1 - 0.94^n)}{1 - 0.94} \\
0.06 &< 1 - 0.94^n \\
-0.4 &< -0.94^n \\
0.4 &> 0.94^n \\
\frac{\ln 0.4}{\ln 0.94} &< n < 14.0 \\
\Rightarrow n &< 14
\end{align*}
\]

Instead of \( > \), \( \leq \) would be better.
8.4 Applications to population growth

- In any given year, the population of any particular group of organisms can change for a variety of reasons. In setting up a population model, it is useful to identify additions to the population (births and immigration) and deletions from the population (deaths and emigration).

- For now, we restrict our attention to situations where the only change in a population is due to births and deaths.

- Let $P_n$ be the population in year $n$. If the population has a yearly birth rate of $b\%$ and death rate of $d\%$, then

\[
P_{n+1} = P_n + \frac{b}{100} P_n - \frac{d}{100} P_n = P_n (1 + \frac{b - d}{100})
\]

- If we are given an initial population, then the above recurrence relation gives us a way to calculate the population change over time.

- Let $P_0$ be the initial population, and let $\frac{b-d}{100} = k$ the population growth rate. Then

\[
P_1 = P_0(1+k)
\]
\[
P_2 = P_1(1+k) = P_0(1+k)^2
\]
\[
P_3 = P_2(1+k) = P_0(1+k)^3
\]
\[
\vdots
\]
\[
P_n = P_{n-1}(1+k) = P_0(1+k)^n
\]

- Thus, for an initial population $P_0$ and growth rate $k$, the population after $n$ years is $P_n = P_0(1+k)^n$.

- This type of population model is called the *exponential population model*. It assumes that the growth rate remains constant, which is not often the case in reality.
Example 8.4.1 The population of a town on January 1, 2003 was 13,690. The birth rate is estimated at 20% and the death rate at 18%. Assuming there is no immigration or emigration from this town, estimate the population on January 1, 2013.

\[ b = \text{birth rate} = 20\% = 0.2 \]
\[ d = \text{death rate} = 18\% = 0.18 \]
\[ k = \text{pop. growth rate} = b - d = 0.02 = 0.02 \]
\[ n = \text{no. of years} = 10. \]
\[ P_0 = 13,690. \]
\[ P_{10} = 13,690 \left(1 + 0.02\right)^{10} \]
\[ \approx 16,688. \]

The population on Jan 1, 2013 will be approximately 16,688.
Example 8.4.2  A colony of penguins in Antarctica numbered 26 in the year 1959 and 160 in the year 1969.

(a) Assuming that this population can be modelled accurately by the exponential population model, determine the annual growth rate of this population.

\[ P_0 = 26 \]
\[ P_{10} = 160 \]
\[ n = 10 \]
\[ k = ? \]

\[ P_n = P_0 \left(1 + k\right)^n \]
\[ 160 = P_0 \left(1 + k\right)^{10} \]
\[ \frac{160}{26} = \left(1 + k\right)^{10} \]
\[ \sqrt[10]{\frac{160}{26}} = 1 + k \]
\[ 1.199 \approx 1 + k \]
\[ k \approx 0.199 \approx 19.9\% \]

(b) Using the growth rate from part (a), predict what the population was in 1979.

\[ P_n = P_0 \left(1 + k\right)^n \]
\[ \text{Predicted population} \approx 160 \left(1 + 0.199\right)^{10} \]
\[ \approx 982 \]

(c) In 1979, the actual penguin population in the colony was 320. List some possible reasons for the difference between your answer to part (b) and the actual population.

- Food (not enough)
- Weather
- Predation
- Human
- Disease
- Migration
- Climate change
Mathematical induction

Let \( P(n) \) be a claim about a natural number \( n \), where \( n \) is greater than or equal to a particular natural number \( n_0 \). If

1. \( P(n_0) \) is true, and

2. if \( P(k) \) is true, then \( P(k + 1) \) is also true, for all natural numbers \( k \geq n_0 \),

then \( P(n) \) is true for all \( n \geq n_0 \).

**Example 8.5.1** Prove that \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \) for all integers \( n \geq 1 \).

Let \( P(n) \) be the claim that \( 1 + 2 + 3 + 4 + \ldots \).

\[
\begin{align*}
\text{LHS} &= \sum_{i=1}^{n} i = 1 + 2 + 3 + 4 + \ldots \\
\text{RHS} &= \frac{n(n+1)}{2} = \frac{1(1)}{2} = 1 \\
\end{align*}
\]

Assume that when \( n = k \), \( \sum_{i=1}^{k} i = \frac{k(k+1)}{2} \).

**INDUCTIVE ASSUMPTION.**

\[
\begin{align*}
\text{LHS} &= \sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) \\
&= \frac{k(k+1) + 2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2} \\
\end{align*}
\]
\[ \text{RHS} = \frac{n(n+1)}{2} \]
\[ = \frac{(h+1)(h+1+1)}{2} \]
\[ = \frac{(h+1)(h+2)}{2} \]
\[ = \frac{h^2 + 3h + 2}{2} \]

\[ \therefore \text{P}(n) \text{ is true by mathematical induction.} \]
Example 8.5.2  Prove that \( \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \), for all integers \( n \geq 1 \).

Let \( P(n) \) be the claim that \( \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \) for \( n \in \mathbb{Z} \), \( n \geq 1 \).

\[
\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}
\]

For \( n = 1 \), \( \sum_{i=1}^{1} i^2 = 1 \cdot (1 + 1)(2 + 1) = 1 \cdot 3 = 3 \), which is true.

**Base Case**

\( n = k+1 \)

**Inductive Step**

\( \sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \)

\[
= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}
\]

\[
= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}
\]

\[
= \frac{(k+1)(2k^2 + 7k + 6)}{6}
\]

\( \therefore P(k+1) \) holds.
\[ \text{NOTES RMS } \quad n = k+1 \]
\[ \frac{(k+1) (k+1+1) (2(k+1)+1)}{6} \]
\[ = \frac{(k+1) (k+2) (2k+3)}{6} \]
\[ = \frac{(k+1) (2k^2 + 3k + 4k + 6)}{6} \]
\[ = \frac{(k+1) (2k^2 + 7k + 6)}{6} \]

\[ \therefore P(k+1) \text{ holds } \Rightarrow P(n) \text{ holds for all integer } n \geq 1. \]
Example 8.5.3  Show that 7 divides $8^n - 1$ for all integers $n \geq 1$.

Let $P(n)$ be the claim that 7 divides $8^n - 1$ for all integers $n \geq 1$.

$n=1$  $8^1 - 1 = 8^1 - 1 = 7 = P(1)$, $P(1)$ holds as 7 is divisible by 7.

$n = k$  Assume $P(k)$ holds, that is, $P(k) = 8^k - 1$ is divisible by 7. That is, $8^k - 1 = 7A$, where A is some integer $7|1$, $A \in \mathbb{Z}, A > 1$.

$n = k + 1$.  WTS that $P(k+1)$ holds.

$P(k+1) = 8^{k+1} - 1$

$= 8 \cdot 8^k - 1$

$= 8 \cdot [8^k - 1 + 8^k]

= 8 \cdot 8^k - 1 + 7$

$= 7(8^k - 1) + 7$

$= 7 \cdot 7A + 7$

where $7A + 7$ is divisible by 7. Therefore, $P(k+1)$ holds.

\[8^{k+1} - 1 = \frac{8^k(8^k - 1) - 9}{9}\]
9 Complex numbers

- Number systems evolved from a basic need to count and to measure. As the need arose to solve more sophisticated problems, less intuitive (but very useful) number systems were introduced.

- Complex numbers evolved from the need to solve quadratic equations that have no solutions in the real number system. To do this, the symbol $i$ was introduced, where $i^2 = -1$. Although this seems artificial (and the term ‘imaginary numbers’ is sometimes used for complex numbers), this breakthrough had a huge impact and complex numbers are widely used in mathematics, physics and engineering.

- We will look at several representations of complex numbers and investigate some important theorems.

- Topics in this section are:
  - Introduction to complex numbers
  - Complex numbers in polar form
  - Powers of complex numbers
  - Fundamental Theorem of Algebra.