5 Vectors

- In modelling the real world, some quantities, such as length, area, time and temperature, can be described by real numbers. We call these \textit{scalar} quantities. However, the description of other quantities, such as \textit{displacement}, \textit{velocity} and \textit{force}, require more than just one real number. They are described by both a \textit{magnitude} and a \textit{direction}. We call these \textit{vector} quantities.

- Vectors can be represented in many ways and have many applications.

- In this section we look at three representations of vectors: geometric form, matrix form and component form.

- Topics in this section are:
  - Introduction to vectors (geometric and matrix form)
  - Addition of vectors
  - Scalar multiplication of vectors
  - Position vectors
  - The norm of a vector
  - Component form of a vector.
5.1 Introduction to vectors

- A vector quantity is something whose specification requires both a magnitude and a direction.

- We normally use bold lower-case letters to represent vectors, or, when handwriting, a lower-case letter with the ~ symbol below it, for example \( \mathbf{v} \).

- Throughout this section we will refer to the \((x, y)\)-plane as 2-space, denoted \( \mathbb{R}^2 \), and \((x, y, z)\)-space as 3-space, denoted \( \mathbb{R}^3 \). All of our vectors can be depicted in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \).

Geometric representation of a vector

- A vector can be represented geometrically in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \) by an arrow.

- The length of the arrow represents the magnitude of the vector, and the direction of the vector is indicated by the direction the arrow is pointing.

- The actual location of the arrow in the diagram is irrelevant, only its magnitude and direction matter.

Example 5.1.1 The vector \( \mathbf{v} \) represents a velocity of 10 km/h in the north-east direction, while the vector \( \mathbf{w} \) represents a velocity of 5 km/h in the north-west direction.
**Example 5.1.2** The three arrows shown below each represent the same vector.

- If $P$ and $Q$ are points in $\mathbb{R}^2$ or $\mathbb{R}^3$, then $\overrightarrow{PQ}$ denotes the vector from $P$ to $Q$. The point $P$ is the tail of the vector and the point $Q$ is the head of the vector.

- Suppose that $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are two representations of the same vector in the $(x, y)$-plane. Let the coordinates of the four points be $P = (x_P, y_P)$, $Q = (x_Q, y_Q)$, $R = (x_R, y_R)$ and $S = (x_S, y_S)$. Since $\overrightarrow{PQ} = \overrightarrow{RS}$, the triangles $PQA$ and $RSB$ are congruent.

Thus

$$PA = RB, \quad \text{so} \quad x_Q - x_P = x_S - x_R,$$

$$AQ = BS, \quad \text{so} \quad y_Q - y_P = y_S - y_R.$$

For every geometric representation of a particular vector $\mathbf{v}$ in $\mathbb{R}^2$, the change in $x$-coordinate is a fixed quantity, and the change in $y$-coordinate is a fixed quantity.
Matrix representation of a vector

- For a (geometric) vector $\mathbf{v} \in \mathbb{R}^2$ with tail at the point $(x_1, y_1)$ and head at the point $(x_2, y_2)$, the matrix form of the vector has 2 rows and 1 column and is written as

\[
\begin{pmatrix}
  x_2 - x_1 \\
  y_2 - y_1
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
\]

- The matrix form of a vector is the same for all geometric representations of the vector.

- The usual notation for writing a general vector $\mathbf{v}$ in matrix form is $\mathbf{v} = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix}$ (a column vector) or $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)$ (a row vector).

- Although we usually write the vector $\mathbf{v}$ as a column vector, we may occasionally write it as a row vector:

- Given a vector $\mathbf{v} = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix}$ in matrix form, you can find a geometric representation of $\mathbf{v}$ by picking any point in the plane as the tail of the vector, moving $\mathbf{v}_1$ units in the $x$-direction and then $\mathbf{v}_2$ units in the $y$-direction to find the point that is the head of the vector.

- For a (geometric) vector $\mathbf{v} = \overrightarrow{PQ}$ in $\mathbb{R}^3$, where $P = (x_P, y_P, z_P)$ and $Q = (x_Q, y_Q, z_Q)$, the matrix form is

\[
\begin{pmatrix}
x_Q - x_P \\
y_Q - y_P \\
z_Q - z_P
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
v_3
\end{pmatrix}
\] or $\mathbf{v} = (v_1, v_2, v_3)$.

- The entries $v_1$ and $v_2$ (or $v_1$, $v_2$ and $v_3$) are called the components of the vector.
Converting from matrix to geometric form

**Example 5.1.3** On the \((x, y)\)-axes below, draw geometric representations of each of the following vectors.

a) \( \mathbf{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \)  
b) \( \mathbf{v} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \)  
c) \( \mathbf{w} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \)

Converting from geometric to matrix form

**Example 5.1.4** Determine the matrix form of each of the vectors drawn below.

\( \mathbf{w} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \) \( \mathbf{v} = \begin{pmatrix} -5 \\ -2 \end{pmatrix} \) \( \mathbf{w} = \begin{pmatrix} -3 \\ 4 \end{pmatrix} \)
5.2 Addition of vectors

Geometric addition of vectors

- Given vectors \( \mathbf{v} \) and \( \mathbf{w} \), we define the sum \( \mathbf{v} + \mathbf{w} \) by the triangle rule for vector addition.

- Let \( P, Q \) and \( R \) be points in \( \mathbb{R}^2 \) (or \( \mathbb{R}^3 \)) such that \( \mathbf{v} = \overrightarrow{PQ} \) and \( \mathbf{w} = \overrightarrow{QR} \). Then \( \mathbf{v} + \mathbf{w} = \overrightarrow{PR} \).

- Note that when adding vectors \( \mathbf{v} \) and \( \mathbf{w} \) geometrically you put the tail of \( \mathbf{w} \) at the head of \( \mathbf{v} \) and then draw the sum \( \mathbf{v} + \mathbf{w} \) from the tail of \( \mathbf{v} \) to the head of \( \mathbf{w} \).

Matrix addition of vectors

- If \( \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \) and \( \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \), then

\[
\mathbf{v} + \mathbf{w} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}.
\]

- Addition of vectors in \( \mathbb{R}^3 \) is the same procedure.
To see that geometric and matrix addition of vectors are equivalent, consider the following diagram showing the addition of vectors \( \mathbf{v} \) and \( \mathbf{w} \).

**Example 5.2.1**

Let \( \mathbf{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \), \( \mathbf{v} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \) and \( \mathbf{w} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} \).

Determine the following vector sums, using matrix addition of vectors and using geometric addition of vectors.

a) \( \mathbf{u} + \mathbf{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -4 \\ 2 \end{pmatrix} \)

   \[ = \begin{pmatrix} -2 \\ 5 \end{pmatrix} \]

b) \( \mathbf{u} + \mathbf{w} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ -3 \end{pmatrix} \)

   \[ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]
- We define the **zero vector** to be the vector (of an appropriate size) with each component equal to zero, and denote it by \( \mathbf{0} \).

\[
\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

- The zero vector has zero magnitude and unspecified direction, so can be represented geometrically as a point.

**Properties of vector addition**

1. Vector addition is **commutative**, that is, \( \mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v} \).

   - To illustrate the commutativity of addition geometrically, consider four points \( P, Q, R \) and \( S \), arranged in 2-space so that

   \[
   \mathbf{v} = \overrightarrow{PQ} = \overrightarrow{SR} \quad \text{and} \quad \mathbf{w} = \overrightarrow{PS} = \overrightarrow{QR}.
   \]

   ![Diagram](https://via.placeholder.com/150)

   Then \( \mathbf{v} + \mathbf{w} = \overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR} = \overrightarrow{PS} + \overrightarrow{SR} = \mathbf{w} + \mathbf{v} \).

2. Vector addition is **associative**, that is

   \[
   \mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}.
   \]

3. \( \mathbf{0} + \mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{v} \).

\[
(2 \div (3 + 4)) = (2 \div 3 + 4)
\]
5.3 Scalar multiplication of vectors

Geometric scalar multiplication of vectors

- Given a vector \( \mathbf{v} \) and a real number \( t \), we define the scalar multiple \( t\mathbf{v} \) to be the vector whose magnitude is \( |t| \) times the magnitude of \( \mathbf{v} \), and whose direction is the same as \( \mathbf{v} \) if \( t > 0 \) and opposite to \( \mathbf{v} \) if \( t < 0 \).

- Note that if \( t = 0 \), then the scalar multiple \( t\mathbf{v} \) is the zero vector.

Matrix scalar multiplication of vectors

- If \( \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \) and \( t \) is a real number, then

\[
 t\mathbf{v} = \begin{pmatrix} t \cdot v_1 \\ t \cdot v_2 \end{pmatrix}
\]

- Scalar multiplication of vectors in \( \mathbb{R}^3 \) is the same procedure.
To see that geometric and matrix scalar multiplication of vectors are equivalent, consider the following diagram showing the multiplication of vector \( \mathbf{v} \) by the scalar \( t \).

\[ t \mathbf{v} \]

\[ \mathbf{v} \]

\[ t \mathbf{v}_2 \]

\[ t \mathbf{v}_1 \]

\[ \mathbf{v}_2 \]

\[ \mathbf{v}_1 \]

---

**Example 5.3.1** Let \( \mathbf{u} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \). Determine the following vector scalar multiples, using matrix scalar multiplication of vectors and using geometric scalar multiplication of vectors.

\[ a) \quad 2\mathbf{u} = 2 \begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} -8 \\ 4 \end{pmatrix} \]

\[ b) \quad -1\mathbf{u} = -1 \begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \]
• Note that we usually write $-1\mathbf{v}$ as $-\mathbf{v}$.

• We can define vector subtraction as a combination of vector addition and scalar multiplication. If $\mathbf{v}$ and $\mathbf{w}$ are two vectors, then

$$\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w}).$$

**Example 5.3.2** Let $\mathbf{u} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$.

Determine the vectors resulting from the following operations, in both matrix form and geometric form.

\[2 - 7 = -5 \quad \quad 2 + (-3) = -1\]

**a)** $\mathbf{u} - \mathbf{v}$

\[
\begin{pmatrix} 2 \\ -1 \end{pmatrix} - \begin{pmatrix} -3 \\ -4 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}
\]

\[
\begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}
\]

**b)** $\mathbf{v} - 2\mathbf{u}$

\[
\begin{pmatrix} -3 \\ -4 \end{pmatrix} - 2\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -7 \\ -2 \end{pmatrix}
\]
5.4 Position vectors

- Of all the geometric representations of a vector $\mathbf{v}$, the one with its tail at the origin is special.

- In $\mathbb{R}^2$, let $P$ be the point with coordinates $(x_P, y_P)$. The vector $\overrightarrow{OP}$ with its tail at the origin $O$ and its head at $P$ is called the position vector of $P$.

- The matrix form of $\overrightarrow{OP}$ is \[
\begin{pmatrix}
1 & 0 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
x_P \\
y_P
\end{pmatrix}
= \begin{pmatrix}
x_P \\
y_P
\end{pmatrix},
\]
which can also be written as $\overrightarrow{OP} = (x_P, y_P)$.

- Similarly, in $\mathbb{R}^3$, let $P$ be the point with coordinates $(x_P, y_P, z_P)$. The vector $\overrightarrow{OP}$ with its tail at the origin $O$ and its head at $P$ is called the position vector of $P$, and the matrix form of $\overrightarrow{OP}$ is \[
\begin{pmatrix}
x_P - 0 \\
y_P - 0 \\
z_P - 0
\end{pmatrix}
= \begin{pmatrix}
x_P \\
y_P \\
z_P
\end{pmatrix},
\]
which can also be written as $\overrightarrow{OP} = (x_P, y_P, z_P)$.

- The coordinates of the point $P$ are the components of the position vector of $P$. 

\begin{center}
\begin{tikzpicture}
\draw[->] (0,0) -- (3,0) node[right] {$x$};
\draw[->] (0,0) -- (0,3) node[above] {$y$};
\draw[->] (0,0) -- (2,2) node[above right] {$P = (x_P, y_P)$};
\end{tikzpicture}
\end{center}
5.5 The norm of a vector

- The *norm* (or *length* or *magnitude*) of the vector $\mathbf{v} = \overrightarrow{PQ}$ is the (shortest) distance between points $P$ and $Q$.

- The norm of the vector $\mathbf{v}$ is denoted $||\mathbf{v}||$.

- In $\mathbb{R}^2$, if $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$, then
  $$\mathbf{v} = \overrightarrow{PQ} = \begin{pmatrix} x_Q - x_P \\ y_Q - y_P \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \text{ and } ||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2}.$$

In $\mathbb{R}^3$, if $P = (x_P, y_P, z_P)$ and $Q = (x_Q, y_Q, z_Q)$, then
  $$\mathbf{v} = \overrightarrow{PQ} = \begin{pmatrix} x_Q - x_P \\ y_Q - y_P \\ z_Q - z_P \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \text{ and } ||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

- Note that for most vectors $\mathbf{v}$ and $\mathbf{w}$, $||\mathbf{v} + \mathbf{w}|| \neq ||\mathbf{v}|| + ||\mathbf{w}||$.

- For any vector $\mathbf{v}$ and any real number $t$, $||t\mathbf{v}|| = |t| ||\mathbf{v}||$. 

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• A vector with norm 1 is called a unit vector.

• The notation \( \hat{\mathbf{v}} \) will be used to denote a unit vector having the same direction as the vector \( \mathbf{v} \).

• For a given vector \( \mathbf{v} \), with norm \( ||\mathbf{v}|| \), the vector

\[
\hat{\mathbf{v}} = \frac{1}{||\mathbf{v}||} \mathbf{v}
\]

is a unit vector in the direction of \( \mathbf{v} \).

**Example 5.5.1** Determine \( \hat{\mathbf{u}} \) and \( \hat{\mathbf{v}} \) in matrix form where

\[
\mathbf{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}.
\]

\[
||\mathbf{u}|| = \sqrt{3^2 + 4^2} = 5
\]

\[
\hat{\mathbf{u}} = \frac{1}{5} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}
\]

(Check \( ||\hat{\mathbf{u}}|| = \sqrt{(3/5)^2 + (4/5)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{1} = 1 \))

\[
||\mathbf{v}|| = \sqrt{(2)^2 + (-1)^2 + 4^2} = \sqrt{4 + 1 + 16} = \sqrt{21}
\]

\[
\hat{\mathbf{v}} = \frac{1}{\sqrt{21}} \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}
\]

\[
\hat{\mathbf{v}} = \frac{1}{\sqrt{21}} \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \frac{1}{\sqrt{21}} \begin{pmatrix} 2 \sqrt{21} \\ -\sqrt{21} \\ 4 \sqrt{21} \end{pmatrix}
\]

\[
= \begin{pmatrix} \frac{2}{2} \\ -\frac{1}{2} \\ \frac{4}{21} \end{pmatrix} = \begin{pmatrix} 2/2 \\ -1/2 \\ 2/21 \end{pmatrix}
\]

\[
= \frac{4}{21} + \frac{-1}{21} + \frac{18}{21} = 1.
\]
5.6 Component form of a vector \( \mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \)

Component form in 2-space

- In the \((x, y)\)-plane, there are two important unit vectors. The unit vector in the direction of the \(x\)-axis is denoted \( \mathbf{i} \), and the unit vector in the direction of the \(y\)-axis is denoted \( \mathbf{j} \), so

\[
\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

- Note that \( \mathbf{i} \) and \( \mathbf{j} \) can also be written as row vectors \( \mathbf{i} = (1, 0) \) and \( \mathbf{j} = (0, 1) \).

- Any vector \( \mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix} \) in \( \mathbb{R}^2 \) can be written as the sum of scalar multiples of \( \mathbf{i} \) and \( \mathbf{j} \), since

\[
\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix} = a \mathbf{i} + b \mathbf{j}.
\]

- The component form of the vector \( \mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix} \) is \( \mathbf{v} = a \mathbf{i} + b \mathbf{j} \).
Component form in 3-space

- In 3-space, there are three important unit vectors. The unit vector in the direction of the x-axis is denoted \( \mathbf{i} \), the unit vector in the direction of the y-axis is denoted \( \mathbf{j} \), and the unit vector in the direction of the z-axis is denoted \( \mathbf{k} \), so

\[
\begin{align*}
\mathbf{i} &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, & \mathbf{j} &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & \text{and} & \mathbf{k} &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\end{align*}
\]

- Note that \( \mathbf{i}, \mathbf{j} \) and \( \mathbf{k} \) can also be written as row vectors \( \mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 0) \) and \( \mathbf{k} = (0, 0, 1) \).

- Any vector \( \mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \) in \( \mathbb{R}^3 \) can be written as the sum of scalar multiples of \( \mathbf{i}, \mathbf{j} \) and \( \mathbf{k} \), since

\[
\mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.
\]

- The component form of the vector \( \mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \) is

\[
\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.
\]

\[
\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} + 0\mathbf{k},
\]

\[
\mathbf{w} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}
\]
Converting vectors from geometric to component form
A vector $v$ in $\mathbb{R}^2$, with magnitude $||v||$ and direction $\theta$ measured anti-clockwise from the positive x-axis, has component form

$$v = ||v||\cos \theta \mathbf{i} + ||v||\sin \theta \mathbf{j}.$$ 

**Example 5.6.1**

a) The vector $v$ in $\mathbb{R}^2$ has magnitude 4 and direction $\frac{2\pi}{3}$. Write $v$ in component form and in matrix form.

$$v = 4 \cos \frac{2\pi}{3} \mathbf{i} + 4 \sin \frac{2\pi}{3} \mathbf{j}$$

$$= 4 \times \left(-\frac{1}{2}\right) \mathbf{i} + 4 \times \left(+\frac{\sqrt{3}}{2}\right) \mathbf{j}$$

$$= -2 \mathbf{i} + 2\sqrt{3} \mathbf{j}$$

b) The vector $w$ in $\mathbb{R}^2$ has magnitude 3 and direction $\frac{3\pi}{2}$. Write $w$ in component form and in matrix form.

$$w = 3 \cos \frac{3\pi}{2} \mathbf{i} + 3 \sin \frac{3\pi}{2} \mathbf{j}$$

$$= 3 \times 0 \mathbf{i} + 3 \times -1 \mathbf{j}$$

$$= \begin{pmatrix} 0 \\ -3 \end{pmatrix}$$
Converting vectors from component to geometric form

A vector \( \mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} \) has magnitude \( ||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2} \). If \( \mathbf{v} \) is non-zero, then its direction \( \theta \), measured anti-clockwise from the positive \( x \)-axis, is obtained as follows.

- Sketch the position vector \( \mathbf{v} \) on a set of \((x, y)\)-axes.
- If \( \mathbf{v} = v_1 \mathbf{i} + 0 \mathbf{j} \) or \( \mathbf{v} = 0 \mathbf{i} + v_2 \mathbf{j} \), then \( \theta \) will be one of 0, \( \frac{\pi}{2} \), \( \pi \) or \( \frac{3\pi}{2} \), and you can determine which it is from the sketch.
- If neither of \( v_1 \) nor \( v_2 \) is zero, then calculate \( \phi = \arctan \left( \frac{v_2}{v_1} \right) \). The value of \( \phi \) will be between 0 and \( \frac{\pi}{2} \).
- The value of \( \theta \) will be one of \( \phi, \frac{\pi}{2} + \phi, \frac{\pi}{2} - \phi \), or \( 2\pi - \phi \). You can identify which it is from your sketch.

**Example 5.6.2**

a) Find the magnitude and direction of the vector \( \mathbf{v} = \mathbf{i} - \sqrt{3} \mathbf{j} \).

\[ ||\mathbf{v}|| = \sqrt{(1)^2 + (-\sqrt{3})^2} \]
\[ = 2 \]
\[ \phi = \arctan \left( \frac{-\sqrt{3}}{1} \right) = \frac{\pi}{3} \]

\[ \mathbf{v} = 2 \cos \left( \frac{\pi}{3} \right) \mathbf{i} + 2 \sin \left( \frac{\pi}{3} \right) \mathbf{j} \]
\[ \Rightarrow \theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3} \]

b) Find the magnitude and direction of the vector \( \mathbf{v} = -2 \mathbf{i} + 3 \mathbf{j} \).

\[ ||\mathbf{v}|| = \sqrt{(-2)^2 + 3^2} \]
\[ = \sqrt{13} \]
\[ \phi = \arctan \left( \frac{3}{-2} \right) \approx 0.98 \text{ radians} \]

\[ \theta \approx \pi - 0.98 \approx 2.16 \text{ radians} \]