4 Integration

- Situations often arise in which we know the derivative of a function and we need to determine the function itself. The reverse process of differentiation is called \textit{antidifferentiation} or \textit{integration}.

- We will start this section by looking at how to determine antiderivatives for some simple functions.

- We will then investigate how to approximate the area of a region under the graph of a function.

- An extremely important theorem, called the Fundamental Theorem of Calculus, tells us how the area under the graph of a function is related to an antiderivative of that function.

- Determining an antiderivative of a function is usually more challenging than determining the derivative of a function. We will take a brief look at the technique of integration by substitution, which can be thought of as the reverse of the chain rule for differentiation.

- Just as the process of differentiation has many practical applications, so does the process of antidifferentiation. We will end this section with a few practical problems.

- Topics in this section are:
  
  - Antiderivatives and indefinite integrals
  - The area under a curve
  - Definite integrals and the Fundamental Theorem of Calculus
  - Integration by substitution
  - Applications of integration.
4.1 Antiderivatives and indefinite integrals

- A function $F$ is called an antiderivative of $f$ on an interval $I$ if $F'(x) = f(x)$ for all $x \in I$.

- For example, $F(x) = 2x^3$ is an antiderivative of $f(x) = 6x^2$ for all $x \in \mathbb{R}$. However, $F(x) = 2x^3 + 10$ is also an antiderivative of $f(x) = 6x^2$ for all $x \in \mathbb{R}$.

- If $F$ is an antiderivative of $f$ on an interval $I$, then the most general antiderivative of $f$ on $I$ is $F(x) + C$ where $C$ is an arbitrary constant.

Rules for antidifferentiation

- Let $F$ be an antiderivative of $f$ and let $k$ be a constant. An antiderivative of $kf(x)$ is $kF(x)$.

- Let $F$ be an antiderivative of $f$ and let $G$ be an antiderivative of $g$. Then $F + G$ is an antiderivative of $f + g$.

Common antiderivatives

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<td>$x^a$ for $a \neq -1$</td>
<td>$\frac{1}{a+1}x^{a+1}$</td>
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<td>$\frac{1}{x}$ for $x &gt; 0$</td>
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Example 4.1.1  Find the most general antiderivative of each of the following functions.

a) \( f(x) = 3x^2 - 7x + 2x^6 \)

\[
F(x) = \frac{x^3}{3} - \frac{7x^2}{2} + \frac{2x^7}{7} + C
\]

\[
F'(x) = 3x - 7x + 2x^6
\]

b) \( f(x) = 4 \sin x \)

\[
F(x) = -4 \cos x + C
\]

\[
F'(x) = -4 \sin x = \omega \cos \omega
\]

- If we are given a particular value of the antiderivative, we can use that information to decide which antiderivative to chose.

Example 4.1.2  Determine the antiderivative \( F \) of \( f(x) = 4 \sin x \) such that \( F(0) = -5 \).

\[
F(x) = -4 \cos x + C
\]

\[
F(0) = -5
\]

\[
-5 = -4 \cos 0 + C
\]

\[
-5 = -4 + C
\]

\[
C = -1
\]

\[
F(x) = -4 \cos x - 1
\]
Example 4.1.3 Suppose that $f$ is a function satisfying $f(1) = 4$ and $f'(x) = 3\sqrt{x} - \frac{1}{x^3}$. Determine $f(x)$.

\[
f'(x) = 3x^{1/2} - x^{-3}
\]

\[
f(x) = \frac{2}{x} + 3x^{3/2} + C
\]

\[
= 2x^{3/2} + \frac{1}{2}x^{-2} + C
\]

\[
= 2x^{3/2} + \frac{1}{2x^2} + C.
\]

\[
f(1) = 4
\]

\[
\Rightarrow 4 = 2 \cdot 1^{3/2} + \frac{1}{2} + C \Rightarrow C = \frac{3}{2}
\]

\[
f(x) = 2x^{3/2} + \frac{1}{2x^2} + \frac{3}{2}
\]

Example 4.1.4 The graph of a function $f$ is shown below. Make a rough sketch of the antiderivative $F$ of $f$ which satisfies $F(0) = 1$. 

MATH1050, 2011. Section 4.
• We need a convenient notation for antiderivatives.

• \( \int f(x) \, dx \) denotes the most general antiderivative of \( f \), and we call this the **indefinite integral** of \( f \).

• Thus, the indefinite integral of \( f \) is defined by

\[
\int f(x) \, dx = F(x) + C
\]

where \( F(x) \) is any antiderivative of \( f \) and \( C \) is an arbitrary constant, called the **constant of integration**.

• The **indefinite integral** of \( f \) gives a family of functions: the antiderivatives of \( f \) as the constant of integration varies.

• After evaluating an indefinite integral, you should always differentiate your result to check your answer.

**Example 4.1.5** Determine the following indefinite integrals.

a) \( \int x^2(2 + x) \, dx \)

\[
= \int (2x^2 + x^3) \, dx
\]

\[
= \frac{2}{3}x^3 + \frac{1}{4}x^4 + C
\]

\[
\left( 0 \right) \sqrt[5]{1} \, u \, du
\]

\[
= \int u^{1/5} \, du
\]

\[
= \frac{5u^{6/5}}{6} + C
\]
• Now consider the general problem of determining the area under the curve defined by \( y = f(x) \), between \( x = a \) and \( x = b \), where \( f(x) \geq 0 \) for \( x \in [a, b] \).

\[
\Delta x = \frac{b - a}{n}.
\]

\[
\frac{4}{2} = \frac{1}{2}
\]

• We subdivide \( A \) into \( n \) strips \( A_1, A_2, \ldots, A_n \) of equal width. The width of each strip is denoted \( \Delta x \), so

\[
\frac{\Delta x}{\text{width}} = \frac{b - a}{n}.
\]

• The strips divide the interval \([a, b]\) into \( n \) subintervals \([x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]\) where \( x_0 = a \) and \( x_n = b \).

• We can then approximate the area \( A \) by summing the areas of \( n \) rectangles, where each rectangle approximates one strip.

• We need to decide on the height of each rectangle. We can do this by evaluating the function at any particular value within each interval. Let \( c_i \) be the value in the \( i \)th subinterval, \( c_i \in [x_{i-1}, x_i] \), that defines the height of the \( i \)th rectangle. Then we approximate \( A \) by the sum

\[
A \approx f(c_1) \Delta x + f(c_2) \Delta x + \cdots + f(c_n) \Delta x.
\]

• This sum is called a Riemann sum and the points we choose to use to determine the height of each rectangle are called sample points.
4.2 The area under a curve

- Consider the problem of finding the area under the curve defined by $f(x) = -x^2 + 7x - 6$ between $x = 2$ and $x = 4$.

- How do we determine the area of a region with a curved side? We start by approximating the area with rectangles.

Using the left side to define the height of the rectangles:

\[
A_L = \frac{1}{2}f(2) + \frac{1}{2}f(2.5) + \frac{1}{2}f(3) + \frac{1}{2}f(3.5)
\]

\[
= \frac{1}{2}(4) + \frac{1}{2}(5.25) + \frac{1}{2}(6) + \frac{1}{2}(6.25)
\]

\[
= 10.75
\]

Using the right side to define the height of the rectangles:

\[
A_R = \frac{1}{2}f(2.5) + \frac{1}{2}f(3) + \frac{1}{2}f(3.5) + \frac{1}{2}f(4)
\]

\[
= \frac{1}{2}(5.25) + \frac{1}{2}(6) + \frac{1}{2}(6.25) + \frac{1}{2}(6)
\]

\[
= 11.75
\]

- The real area is between 10.75 and 11.75. By taking more rectangles, we can get a better approximation to the area.
• In practice we usually choose either the left-hand endpoints, or the right-hand endpoints, of each subinterval as the sample points.

• If we use the left-hand endpoints of each subinterval to determine the height of the corresponding rectangle, then the Riemann sum becomes

\[
A_L = f(x_0) \Delta x + f(x_1) \Delta x + \cdots + f(x_{n-1}) \Delta x
= \sum_{i=1}^{n} f(x_{i-1}) \Delta x.
\]

• If we use the right-hand endpoints of each subinterval to determine the height of the corresponding rectangle, then the Riemann sum becomes

\[
A_R = f(x_1) \Delta x + f(x_2) \Delta x + \cdots + f(x_n) \Delta x
= \sum_{i=1}^{n} f(x_i) \Delta x.
\]

• It can be shown that the limit of a Riemann sum, as \( n \to \infty \), is independent of the sample points chosen.

• We define the area \( A \) under the curve to be this limit. Thus

\[
A = \lim_{n \to \infty} A_L = \lim_{n \to \infty} A_R.
\]
Example 4.2.1  Determine the Riemann sum for \( f(x) = -x^2 + 7x - 6 \), with \( 2 \leq x \leq 4 \), having 10 subintervals and taking the sample points to be the right endpoints.

You can use the following facts:

\[
\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \\
\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}
\]

\[\Delta x = \frac{4-2}{10} = \frac{1}{5}\]

\[A = \frac{1}{5} \times 4(f(2\frac{1}{5})) + \frac{1}{5} \times \frac{1}{4}(2\frac{2}{5}) + \frac{1}{5} \left( f\left(2\frac{3}{5}\right) \right) + \ldots + \frac{1}{5} f\left(4\frac{1}{5}\right)\]

\[= 11 \frac{13}{25}\]

continued...
The limit of a Riemann sum of a function $f$ over the interval $[a, b]$ gives the area under the curve $f$ only if the curve lies above the $x$-axis for all $x \in [a, b]$.

- If we evaluate a Riemann sum of a function $f$ over an interval on which $f$ is not always positive, then the rectangles below the $x$-axis count as negative area.

**Example 4.2.2** Let $R_n$ be a Riemann sum of $f(x) = \sin x$ over the interval $[a, b]$ having $n$ rectangles.

a) If $[a, b] = [0, 2\pi]$, will $\lim_{n \to \infty} R_n$ be positive, negative or zero.

b) If $[a, b] = [0, \frac{3\pi}{2}]$, will $\lim_{n \to \infty} R_n$ be positive, negative or zero.

c) If $[a, b] = [\frac{\pi}{2}, 2\pi]$, will $\lim_{n \to \infty} R_n$ be positive, negative or zero.
4.3 Definite integrals and the Fundamental Theorem of Calculus

- Let $R_n$ be a Riemann sum for a continuous function $f$ over the interval $a \leq x \leq b$, having $n$ subintervals. Then the definite integral of $f$ from $a$ to $b$ is

$$\int_a^b f(x) \, dx = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x$$

- The definite integral is also called the Riemann integral.

- We call $a$ and $b$ the limits of integration where $a$ the lower limit and $b$ the upper limit.

- The integral sign $\int$ was introduced by Leibniz. It is an elongated S and was chosen because the integral represents the limit of a sum.

- Note that the definite integral is a number. Thus

$$\int_a^b f(x) \, dx = \int_a^b f(u) \, du = \int_a^b f(r) \, dr.$$

Properties of the definite integral

- $\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx$

- $\int_a^a f(x) \, dx = 0$

- $\int_a^b c \, dx = c(b-a)$ where $c$ is a constant

- $\int_a^b [f(x) + g(x)] \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx$

- $\int_a^b cf(x) \, dx = c \int_a^b f(x) \, dx$ where $c$ is a constant

- $\int_a^b f(x) \, dx + \int_a^b f(x) \, dx = \int_a^b f(x) \, dx$
The Fundamental Theorem of Calculus
If $f$ is continuous on $[a, b]$, then

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

where $F$ is any antiderivative of $f$ on $[a, b]$.

Let $A(b)$ denote the area under the curve $y = f(x)$ from $x = a$ to $x = b$. From the diagram we see that

$$f(b) \approx \frac{A(b + \Delta x) - A(b)}{\Delta x}$$

As we take the limit $\Delta x \to 0$ we obtain

$$f(b) = \lim_{\Delta x \to 0} \frac{A(b + \Delta x) - A(b)}{\Delta x} = A'(b).$$

This shows us that the area $A$ is an antiderivative of the function $f$. We have also seen that the area is given by the Riemann integral, so we have

$$A(b) = \int_a^b f(x) \, dx = F(b) + C$$

where $F(x)$ is an antiderivative of $f(x)$ and $C$ is some constant of integration.
Next we observe that when $b = a$ the area must be zero, which allows to determine that

$$C = -F(a)$$

and finally

$$\int_a^b f(x) \, dx = F(b) - F(a).$$

- Thus, for functions for which we can determine an antiderivative, the time-consuming process of calculating definite integrals using sums and limits can be replaced by determining and evaluating an antiderivative.

- The notation $[F(x)]_a^b$ is often used for $F(b) - F(a)$.

- This is a simplified version of the FTC. If you continue your studies in mathematics, you will encounter it in more detail.

**Example 4.3.1** Determine the area under the curve $f(x) = -x^2 + 7x - 6$ between $x = 2$ and $x = 4$. 

$$\int_2^4 f(x) \, dx = \int_2^4 (-x^2 + 7x - 6) \, dx$$

$$= \left[ -\frac{x^3}{3} + \frac{7x^2}{2} - 6x \right]_2^4$$

$$= \left( -\frac{64}{3} + \frac{7 \cdot 16}{2} - 24 \right) - \left( -\frac{8}{3} + \frac{7 \cdot 4}{2} - 12 \right)$$

$$= \left( -\frac{64 + 160 - 72}{3} - \left( -\frac{8 + 28 - 36}{3} \right) \right)$$

$$= \frac{32}{3} - \frac{-2}{3} = \frac{34}{3}$$

$$= 11 \frac{1}{3}.$$
Example 4.3.2  Refer back to Example 4.2.2.

(a) Determine \( \int_0^{2\pi} \sin x \, dx \).

\[
\int_0^{2\pi} \sin x \, dx = \left[ -\cos x \right]_0^{2\pi} = -\cos 2\pi - (-\cos 0) = -1 - (-1) = 0.
\]

(b) Use the properties of the definite integral to determine the area enclosed by the sine curve and the \( x \)-axis over the interval \([0, 2\pi]\).

\[
A = \left| \int_0^{\pi} \sin x \, dx \right| + \left| \int_{\pi}^{2\pi} \sin x \, dx \right|
\]

optional \[
= \left[ -\cos x \right]_0^{\pi} + \left[ -\cos x \right]_\pi^{2\pi}
\]

\[
= -\cos \pi - (-\cos 0) + \left( -\cos 2\pi - -\cos \pi \right) = -1 + 1 + 1 - 1 = 2 + 2 = 4.
\]

---

Example 4.3.3  Determine \( \int_{-1}^{1} \frac{1}{x^2} \, dx \).

\[
\int_{-1}^{1} \frac{1}{x^2} \, dx = \int_{-1}^{1} x^{-2} \, dx = \left[ -x^{-1} \right]_1^{-1} = -1 - 1 = 0.
\]

---

\( f(x) \) is not \( \text{ctv} \) over \([-1, 1] \): \( \int_{-1}^{1} f(x) \, dx \) does not exist.
4.4 Integration by substitution

- Recall the Chain rule for differentiation: If \( f \) and \( g \) are differentiable functions, then 
  \[
  (f \circ g)'(x) = f'(g(x))g'(x).
  \]
- Reversing the Chain rule gives the following integration formula:
  
  \[
  \int f'(g(x))g'(x) \, dx = f(g(x)) + C.
  \]

**Example 4.4.1** Determine \( \int 2xe^{x^2} \, dx \).

\[
\begin{align*}
  &\text{Let } u = x^2, \quad y = e^u \\
  &\quad \frac{dy}{du} = e^u, \quad \frac{dx}{du} = 2x \\
  &\quad \int 2xe^{x^2} \, dx = \int e^u \, (2x \, du) = e^{x^2} + C
\end{align*}
\]

- Sometimes it is hard to see that an integrand has the form \( f'(g(x))g'(x) \), so we need a systematic way to apply the above integration formula.
- If \( y = f(x) \), where \( f \) is a differentiable function, then the differentials \( dx \) and \( dy \) are related by the equation 
  \[
  dy = f'(x) \, dx.
  \]
- The geometric meaning of the differentials is that if \( \ell \) is the tangent line to \( f \) at the point \( P = (x, f(x)) \), then the point \( R = (x + dx, f(x) + dy) \) is also on the line \( \ell \).
- It can be shown that you can interpret the \( dx \) in an indefinite integral as a differential. Thus, if \( u = g(x) \), then 
  \[
  du = g'(x) \, dx
  \]

\[
\begin{align*}
  \int 2x \, dx & = 2x + C \\
  \int 2y \, dx & = 2y^2 + C \\
  \int 2 \, dx & = x + C
\end{align*}
\]
Integration by substitution

If \( u = g(x) \) is a differentiable function whose range is an interval \( I \) and \( f \) is continuous on \( I \), then

\[
\int \frac{f(g(x))g'(x)}{g(x)} \, dx = \int f(u) \, du = F(u) + C = F(g(x)) + C
\]

where \( F \) is an antiderivative of \( f \).

**Example 4.4.2** Use integration by substitution to determine the following indefinite integrals.

a) \( \int 2xe^{x^2} \, dx \)

\[
\text{let } u = x^2 \quad \int 2xe^{x^2} \, dx = \int 2x \cdot e^u \, du \quad = \int 2e^u \, du = e^u + C \quad = e^{x^2} + C
\]

b) \( \int (6x^2 - 2)^{\frac{3}{2}} \, dx \)

\[
= \left( 5x^2 - 2 \right)^{\frac{5}{2}} + C
\]

\[
\int \frac{4}{(3x^4 - 2)^{\frac{3}{4}}} \, dx \quad = \quad 4(3x^4 - 2)^{-\frac{3}{4}} \cdot \frac{1}{4} 
\]

\[
= \quad 30x(3x^2 - 2)^{\frac{4}{4}} 
\]

\[
= \quad 30x(3x^2 - 2)^{\frac{3}{2}} 
\]

\[
= \quad 30x \cdot 6 
\]

\[
= \quad 180x 
\]

\[
\int 2x \sin(4x^3 + 2) \, dx \]

\[
\text{let } u = 4x^3 + 2 \quad \int 2x \sin (u) \, du = \int 2x \sin u \, du = \int \sin u \, du = -\cos u + C 
\]

\[
= \frac{1}{6} \cos (4x^3 + 2) + C = -\frac{\cos (4x^3 + 2)}{6} + C
\]
• It can be difficult to choose a good substitution. If the first function you choose doesn't seem to work, try something else. The key is to look for a function that appears in the integrand AND whose derivative also appears in the integrand (perhaps multiplied by a constant).

• Sometimes there is more than one substitution that will work.

**Example 4.4.3** Determine the following indefinite integrals by applying the indicated substitution.

a) \[ \int \sqrt{3x - 2} \, dx \text{ using } u = 3x - 2 \]

\[ \frac{du}{dx} = 3 \implies \frac{du}{3} = dx \]

\[ \int \frac{u^{1/2}}{3} \, du = \frac{2}{3} u^{3/2} + C \]

b) \[ \int \sqrt{3x - 2} \, dx \text{ using } u = \sqrt{3x - 2} \]

\[ u = (3x - 2)^{1/2} \]

\[ \frac{du}{dx} = \frac{1}{2} (3x - 2)^{-1/2} \cdot 3 \]

\[ \int \frac{\sqrt{3x - 2}}{3} \, du = \frac{3}{2} \sqrt{3x - 2} + C \]
- Since the function $f(x) = \ln x$ has domain $(0, \infty)$, we need to ensure that we never take the natural logarithm of a negative number or zero.

- If $f(x) = \ln x$, where $x > 0$, then
  
  $f'(x) = \frac{1}{x}$
  
  and so
  
  $\int \frac{1}{x} \, dx = \ln x + C$.

  If $f(x) = \ln(-x)$ where $x < 0$, then we have

  $f'(x) = \frac{1}{-x} \times (-1) = \frac{1}{x}$
  
  and so
  
  $\int \frac{1}{x} \, dx = \ln(-x) + C$.

  We often combine these cases and write

  $\int \frac{1}{x} \, dx = \ln|x| + C$, for $x \neq 0$.

- An integral of the form

  $\int \frac{g'(x)}{g(x)} \, dx$

  can usually be treated by integration by substitution.

- Let $u = g(x)$, so $du = g'(x) \, dx$. Then

  $\int \frac{g'(x)}{g(x)} \, dx = \int \frac{1}{u} \, du = \ln|u| + C = \ln|g(x)| + C$. 

  \[ \frac{d}{dx} \ln x = \frac{1}{x} \]

  \[ \ln(2x) = \frac{2}{2x} = \frac{1}{x} \]
Example 4.4.4  Determine the following indefinite integrals

a) \( \int \frac{2x}{x^2 + 3} \, dx \)  
   Let \( u = x^2 + 3 \)
   \[
   du = 2x \, dx \Rightarrow dx = \frac{du}{2x}
   \]
   \[
   \int \frac{2x}{x^2 + 3} \, dx = \int \frac{2x}{u} \cdot \frac{du}{2x} = \int \frac{1}{u} \, du = \ln |u| + C
   \]
   \[
   = \ln |x^2 + 3| + C
   \]

b) \( \int \frac{6x^2 - 4x}{x^3 - x^2} \, dx \)  
   Let \( u = x^3 - x^2 \)
   \[
   du = 3x^2 - 2x \, dx
   \]
   \[
   = \int \frac{6x^2 - 4x}{u} \cdot \frac{du}{(3x^2 - 2x)} = \int \frac{2}{u} \, du = 2 \ln |u| + C
   \]
   \[
   = 2 \ln |x^3 - x^2| + C
   \]

c) \( \int -\tan x \, dx \)
   \[
   = \int -\frac{\sin x}{\cos x} \, dx
   \]
   Let \( u = \cos x \)
   \[
   du = -\sin x \, dx
   \]
   \[
   = \int \frac{-\sin x}{u} \cdot \frac{du}{-\sin x} = \int \frac{1}{u} \, du = \ln |u| + C
   \]
   \[
   = \ln |\cos x| + C
   \]
Integration by substitution for definite integrals

- There are two ways to apply integration by substitution to a definite integral.

- The first way is to first determine the indefinite integral using integration by substitution and then use this to solve the definite integral.

**Example 4.4.5** Evaluate

\[
\begin{align*}
\int_{1}^{2} 18x^4(10 - 3x^2)^3 \, dx & \quad \text{(x bounds)} \\
\int_{1}^{2} & (10 - 3x^2) \, dx \\
\int_{1}^{2} & 10 \, dx - 3 \int_{1}^{2} x^2 \, dx \\
& = 10x \bigg|_{1}^{2} - 3 \left( \frac{x^3}{3} \right) \bigg|_{1}^{2} \\
& = 10(2 - 1) - 3 \left( \frac{2^3}{3} - \frac{1^3}{3} \right) \\
& = 10 - 3 \left( \frac{8 - 1}{3} \right) \\
& = 10 - 3 \left( \frac{7}{3} \right) \\
& = 10 - 7 \\
& = 3
\end{align*}
\]

**Example 4.4.5** Evaluate

\[
\begin{align*}
\int_{1}^{2} & 18x^4(10 - 3x^2)^3 \, dx \\
& = x^4 \int_{1}^{2} (10 - 3x^2)^3 \, dx \\
& = x^4 \left[ -3 \int 10 - 3x^2 \, du \right] \\
& = x^{14} \left[ -3 \left( \frac{10 - 3x^2}{4} \right)^4 \right]^{2} _{1} \\
& = x^{14} \left( \frac{10 - 3(2)^2}{4} - \frac{10 - 3(1)^2}{4} \right) \\
& = x^{14} \left( \frac{10 - 12}{4} \right) - \frac{7203}{4} \\
& = \frac{3 \times 7^4}{4} - \frac{3 \times 12^4}{4} \\
& = \frac{-7203 + 48}{4} \\
& = \frac{-7155}{4} \\
& = -1788.75
\end{align*}
\]
• The second way is to change the limits of integration when the substitution is performed.

• If \( g' \) is continuous on \([a, b]\) and \( f \) is continuous on the range of \( u = g(x) \), then

\[
\int_{a}^{b} f(g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du.
\]

**Example 4.4.6** Evaluate the following definite integrals.

a) \( \int_{-2}^{1} 18x(10 - 3x^2)^3 \, dx \)

Let \( u = 10 - 3x^2 \)
\[
\frac{du}{dx} = -6x
\]

\( x = -2 \quad u = -2 \)
\( x = 1 \quad u = 7 \)

\[
\int_{-2}^{1} 18x(10 - 3x^2)^3 \, dx = \int_{-2}^{7} -3u^3 \, du
\]

\[
= \left[ -\frac{3u^4}{4} \right]_{-2}^{7}
\]

\[
= -\frac{3 \cdot 7^4}{4} - \left( -\frac{3 \cdot (-2)^4}{4} \right)
\]

\[
= -\frac{7843}{4}
\]

continued...
Example 4.4.6 (continued)

b) \[ \int_0^7 \frac{9\sqrt{4 + 3x}}{4 + 3x} \, dx \]

Let \( u = 4 + 3x \)
\[ du = 3 \, dx \]
\[ dx = \frac{du}{3} \]

\[ \int_0^7 9u^{\frac{1}{2}} \, du \]
\[ = \left[ \frac{9}{3} 2u^{\frac{3}{2}} \right]_0^7 \]
\[ = \frac{9}{3} (2(4 + 3 \cdot 7)^{\frac{3}{2}} - 2(4)^{\frac{3}{2}}) \]
\[ = 2 \left( 25^{\frac{3}{2}} - 4^{\frac{3}{2}} \right) \]
\[ = 2 \left( 5 \cdot 25 - 2 \cdot 4 \right) \]
\[ = 234 \text{, sq. m}^3 \]
Proof of the substitution rule for definite integrals

We need to prove that if \( g' \) is continuous on \([a, b]\) and \( f \) is continuous on the range of \( u = g(x) \), then

\[
\int_a^b f(g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du. \tag{1}
\]

Let \( F \) be an antiderivative of \( f \). Then by the substitution rule for indefinite integrals, \( F(g(x)) \) is an antiderivative of \( f(g(x))g'(x) \).

By applying the FTC to the left-hand side of (1) above,

\[
\int_a^b f(g(x))g'(x) \, dx = [F(g(x))]_a^b = F(g(b)) - F(g(a)).
\]

By applying the FTC to the right-hand side of (1) above,

\[
\int_{g(a)}^{g(b)} f(u) \, du = [F(u)]_{g(a)}^{g(b)} = F(g(b)) - F(g(a)).
\]

Thus

\[
\int_a^b f(g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du.
\]
4.5 Applications of integration

- When faced with a practical problem in which we know the rate at which a quantity is changing, integration can give us information about that quantity.

- For a particle moving in a straight line, an antiderivative of the acceleration function \( a(t) \) is the velocity function \( v(t) \) and an antiderivative of the velocity function \( v(t) \) is the displacement function \( s(t) \).

- Additional information about the velocity or displacement at a particular time can help us to choose the correct antiderivative.

**Example 4.5.1** A balloon filled with water is thrown upward from Level 6 of the outside stairs of the Priestley building, with an initial speed of 10 m/s. The balloon is thrown from an initial height of 20 metres. Assume that the acceleration due to gravity is 9.8 m/s\(^2\) (downward).

a) Determine a function \( v(t) \) describing the velocity of the balloon at \( t \) seconds.

b) Determine a function \( s(t) \) describing the displacement of the balloon at \( t \) seconds.

c) When does the balloon reach its maximum height and what is this maximum height?

d) When does the balloon hit the ground and what is its velocity just before it hits the ground?

\[ s(t) = \begin{cases} 1 & -\infty < t < 0 \\ 0 & t \geq 0 \end{cases} \]

continued...

\[ v(t) = \begin{cases} 1 & -\infty < t < 0 \\ 0 & t \geq 0 \end{cases} \]
Example 4.5.1 (continued)

a) \[ \int a(t) \, dt = v(t) + C \]

\[ v(t) = \int -9.8 \, dt \]

\[ = -9.8t + C \]

\[ v(0) = 10 \Rightarrow 10 = -9.8 \times 0 + C \]

\[ \therefore C = 10 \]

\[ \therefore v(t) = -9.8t + 10 \]

b) \[ S(t) = \int v(t) \, dt \]

\[ = \int (-9.8t + 10) \, dt \]

\[ = -4.9t^2 + 10t + D \]

\[ S(0) = 20 \]

\[ 20 = -4.9 \times 0^2 + 10 \times 0 + D \]

\[ D = 20 \]

\[ \therefore S(t) = -4.9t^2 + 10t + 20 \]

c) Max height \[ \Rightarrow \quad v(t) = 0. \]

\[ -9.8t + 10 = 0 \]

\[ t = \frac{10}{9.8} \approx 1 \text{ sec.} \]

\[ S(1) \approx -4.9 \times 1^2 + 10 \times 1 + 20 \approx 25.1 \text{ m.} \]
a) \( s(t) = 0 \)

\[ 0 = -49t^2 + 10t + 20. \]

Using the quadratic formula,

\[ t \approx 3.2 \text{ sec} \]

(ignoring \( t = -ve \) because you can't have a negative time).

\[ u(3.2) \approx -9.8 \times 3.2 + 10 \]

\[ \approx 20 \text{ m/s}. \]
**Example 4.5.2** The velocity of a car driving along a straight road is described by the following graph. By approximating this curve using straight line segments, estimate the total distance travelled by the car in 5 seconds.

\[
A_1 = \frac{1 \times 10}{2} = 5
\]
\[
A_2 = 1 \times 10 = 10
\]
\[
A_3 = 10
\]
\[
A_4 = A_1 = 5
\]
\[
A_5 = 1 \times 20 = 20
\]
\[
A_6 = \frac{1 \times 20}{2} = 10
\]

Total area = 60

Distance is the antiderivative of velocity (as I'm travelling in a straight line, not backtracking), so my total distance travelled is \( \approx 60 \text{m} \).
- It is important to distinguish between the displacement of an object at time $t$ and the total distance travelled by the object at time $t$.

- When an object moves in a straight line and its velocity changes from positive to negative (or negative to positive) then the object is backtracking on ground already covered, so displacement is reduced but total distance continues to accumulate.

- Displacement is given by a definite integral but total distance travelled is given by the area enclosed by the velocity curve and the horizontal axis (time).

- If an object moving in a straight line has displacement function $s(t)$ and velocity function $v(t) = s'(t)$, then

$$\int_{t_1}^{t_2} v(t) \, dt = s(t_2) - s(t_1)$$

is the net change of displacement (position) of the object during the time period from $t_1$ to $t_2$.

- If the object is moving in the same direction for all of the time interval $t_1$ to $t_2$, then this is also the total distance travelled during that time.
Example 4.5.3 A particle moves along a line with acceleration at time $t$ of $a(t) = 2t + 2$ m/s$^2$. The particle has an initial velocity of $-3$ m/s.

a) Determine the displacement of the particle after 2 seconds.

\[ u(t) = \int a(t) \, dt \]
\[ = \int (2t+2) \, dt \]
\[ = t^2 + 2t + C. \]

\[ u(0) = -3 \]
\[ \Rightarrow -3 = 0^2 + 0 + C \]
\[ \Rightarrow C = -3 \]
\[ \therefore u(t) = t^2 + 2t - 3 \]

\[ S(t) = \int u(t) \, dt \]
\[ = \int (t^2 + 2t - 3) \, dt \]
\[ = \frac{t^3}{3} + t^2 - 3t + D. \]

Displacement after 2 sec
\[ = \left[ \frac{t^3}{3} + t^2 - 3t + D \right]_0^2 = \frac{8}{3} + 4 - 6 = \frac{2}{3} \text{ m.} \]

continued...
Example 4.5.3 (continued)  

b) Determine the total distance the particle travels in the first two seconds.

\[ \begin{align*}
  v(t) &= t^2 + 2t - 3 \\
  S(t) &= \frac{t^3}{3} + t^2 - 3t + 1 \\
  S(0) &= 0 \\
  \Rightarrow S(t) &= \frac{t^3}{3} + t^2 - 3t
\end{align*} \]

\[ \begin{align*}
  v(t) &= t^2 + 2t - 3 \\
  0 &= (t + 3)(t - 1) \\
  t &= -3, 1
\end{align*} \]

Negative displacement, so I need to calculate the area under the velocity curve from \( t = 0 \) to 2.

\[ \begin{align*}
  0 &= \int_{0}^{1} (t^2 + 2t - 3) \, dt \\
  &\quad + \int_{1}^{2} (t^2 + 2t - 3) \, dt
\end{align*} \]
Example 4.5.4 Water flows from the bottom of a storage tank at a rate of \( r(t) = -6t + 180 \) litres per minute, where \( 0 \leq t \leq 30 \).

(a) Find the amount of water that flows from the tank during the first 10 minutes.

Let amount = \( A(t) \)

\[
A(t) = \int r(t) \, dt \\
= \int (-6t + 180) \, dt \\
= -3t^2 + 180t + C
\]

Amount out in 1st 10 mins. = \([-3t^2 + 180t + C]\)\(_{0}^{10}\)

\[
= -3 \times 10^2 + 1800 = 1500
\]

(b) If the tank empties in 30 minutes, determine how much water was in the tank at time \( t = 0 \).

\[
A(30) = \left[ -3t^2 + 180t + C \right]_{0}^{30} \\
= -3 \times 30^2 + 180 \times 30 \\
= -2700 + 5400 \\
= 2700 \text{ L}
\]

\[\therefore \text{There was } 2700 \text{ L in tank at } t = 0.\]
Distance = \left| \int_0^1 (t^3 + 2t - 3) \, dt \right| + \int_1^2 (t^2 + 2t - 3) \, dt
\begin{align*}
&= \left| \left[ \frac{t^3}{3} + t^2 - 3t \right]_0^1 + \left[ \frac{t^3}{3} + t^2 - 3t \right]_1^2 \right| \\
&= \left| (\frac{1}{3} + 1 - 3) - 0 \right| + \left( \frac{8}{3} + 4 - 6 \right) - \left( \frac{1}{3} + 1 - 3 \right) \\
&= \left| -\frac{4}{3} - \frac{2}{3} \right| + \frac{2}{3} - \frac{2}{3} \\
&= \frac{2}{3} + \frac{2}{3} + \frac{2}{3} \\
&= \frac{4}{3} \text{ m}
\end{align*}