3 Differentiation

- In many applications of mathematics, it is very important to be able to calculate the gradient, or slope, of the graph of a function at a particular point.

- This gradient is called the derivative of the function at the point and it measures the rate of change of the function at that point.

- Derivatives of functions provide useful information about the graph of a function and also have important applications in many areas of science and economics.

- In this section we will look at the definition of the derivative of a function and review some rules that allow us to calculate the derivatives of many functions.

- We will apply our knowledge of derivatives to sketch the graphs of functions and to solve optimisation problems and problems involving rates of change.

- Topics in this section are:
  
  - Tangent lines
  - The derivative of a function
  - Differentiation rules
  - Critical points and curve sketching
  - Applications of differentiation.

\[ \frac{dy}{dx} = ? \]
3.1 Tangent lines

- The slope of a line describes the rate of change of $y$ with respect to $x$.

\[ m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \]

Rapid increase  Slow increase  Slow decrease  Rapid decrease

- A secant line is a line that passes through two points on a curve.

- For a curve $y = f(x)$, we can use the slope of a secant line to describe the average rate of change of $f(x)$ with respect to $x$ over a given interval of values for $x$.

- The slope of the secant line passing through the points $(a, f(a))$ and $(b, f(b))$ is $\frac{f(b) - f(a)}{b - a}$.

- Consider the family of secant lines that we obtain by letting $b$ get closer and closer to $a$.

- The limit of these lines (as $b \to a$) is the line that just touches the curve at the point $(a, f(a))$. We call this line the tangent line to $f(x)$ at the point $x = a$.

- The slope of the tangent line to $f(x)$ at the point $x = a$ is $\lim_{b \to a} \frac{f(b) - f(a)}{b - a}$.
The slope of the tangent line at \( x = a \) gives the \textit{instantaneous rate of change} of \( f(x) \) with respect to \( x \) at that point.

\textbf{Example 3.1.1} We can guess the slope of the tangent line to the curve \( f(x) = 2x^2 - x + 1 \) at \( x = 1 \) by evaluating the slopes of the secant lines passing through \( (1, f(1)) \) and \( (b, f(b)) \) for values of \( b \) that approach 1.

\[
\begin{array}{c|c|c|c|c}
 b & 1.5 & 1.1 & 1.01 & 1.001 \\
\hline
\frac{f(b) - f(1)}{b - 1} & \frac{4 - 2}{1.5 - 1} & \frac{2.32 - 2}{1.1 - 1} & \frac{2.0302 - 2}{1.01 - 1} & \frac{2.003002 - 2}{1.001 - 1} \\
\end{array}
\]

\[
\frac{4 - 2}{1.5 - 1} = 4, \quad \frac{2.32 - 2}{1.1 - 1} = 3.2, \quad \frac{2.0302 - 2}{1.01 - 1} = 3.02, \quad \frac{2.003002 - 2}{1.001 - 1} = 3.002
\]

We might guess that the slope of the tangent line at \( x = 1 \) is 3.

Determine the slope of the tangent line to the curve \( f(x) = 2x^2 - x + 1 \) at \( x = 1 \) by evaluating the limit

\[
\lim_{b \to 1} \left( \frac{2b^2 - b + 1}{b - 1} \right) = \lim_{b \to 1} \left( \frac{2b^2 - b - 1}{b - 1} \right) = \lim_{b \to 1} \left( \frac{2b + 1}{b + 1} \right) = 2x + 1 = 3.
\]
Another way of writing the slope of the tangent line to \( f(x) \) at \( x = a \) is to let \( b = a + h \). Then \( b \to a \) is equivalent to \( h \to 0 \) and the slope of the tangent line to \( f(x) \) at \( x = a \) is

\[
\frac{f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{a + h - a}}{\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}}.
\]

**Example 3.1.2**

**a)** Determine the slope of the tangent line to the curve \( f(x) = x^3 - 2x \) at \( x = 2 \).

\[
\lim_{h \to 0} \frac{(a+h)^3 - 2(a+h) - (a^3 - 2a)}{h} = a = 2
\]

\[
\lim_{h \to 0} \frac{(a^3 + 2ah + h^2)(a+h) - 2a - 2h - a^3 + 2a}{h} = 10
\]

\[
\lim_{h \to 0} \frac{3a^2h + 3ah + h^3 - 2h}{h} = \lim_{h \to 0} \frac{3ah^2 + h^3 - 2h}{h} = 10
\]

**b)** Use your answer from part (a) to determine the equation of the tangent line to the curve \( f(x) = x^3 - 2x \) at \( x = 2 \).

\[
y = mx + c
\]

\[
m = 10
\]

\[
x = 2
\]

\[
y = 4(2^3 - 4) + c
\]

\[
y = 10x - 16.
\]
3.2 The derivative of a function

- The derivative of a function \( f \) at the number \( a \), denoted by \( f'(a) \) is

\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
\]

if this limit exists.

- Thus the tangent line to the curve \( y = f(x) \) at \( x = a \) is the line through \((a, f(a))\) whose slope is equal to \( f'(a) \), the derivative of \( f \) at \( a \).

- The derivative \( f'(a) \) is the instantaneous rate of change of \( y = f(x) \) with respect to \( x \) when \( x = a \).

**Example 3.2.1** In a controlled laboratory experiment, the number of bacteria after \( t \) hours is described by a function \( n(t) \) having the following graph.

![Graph of bacteria growth]

a) What does the derivative \( n'(t) \) represent?

*growth of bacteria has quickly the*

b) On the graph, locate the value of \( t \) that gives the largest value of the derivative.  

blue line
• Given a function $f$, we can define a new function, called the derivative of $f$, and denoted $f'$, by the rule that

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \]

The domain of $f'$ is the set of all $x$ in the domain of $f$ such that $f'(x)$ exists.

• The derivative of $f$ is also called the derived function of $f$. The process of determining the derivative of a function $f$ is called differentiation.

**Example 3.2.2** Use the definition of the derivative to determine the derived function of $f$ where $f(x) = \sqrt{x}$. Draw the graphs of $f$ and $f'$.

\[
\begin{align*}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.
\end{align*}
\]
Example 3.2.3  The graph of a function $f$ is shown below. Sketch (roughly) the graph of the derivative of $f$.

- If $y = f(x)$, then other notation commonly used for the derivative $f'(x)$ includes:
  \[ y' \quad \frac{dy}{dx} \quad \frac{df}{dx} \quad \frac{d}{dx}f(x). \]

- A function $f$ is said to be differentiable at $a$ if $f'(a)$ exists. Note that a function may not be differentiable at all points in its domain.

\[ m = \frac{dy}{dx} \]

\begin{align*}
  &\text{not smooth at } x = 0 \\
  &\text{not continuous at } x = 1 \\
  &\text{too steep at } x = 0
\end{align*}

\[ y = |x|. \]
3.3 Differentiation rules

- Determining derivatives from the definition can be time-consuming. Luckily there are some rules for differentiation that speed up the process. The proofs of these rules can be found in most calculus textbooks, but you do not need to know the proofs for this course.

- The derivative of a constant function is zero:
  \[ y = 3x + 6 \]
  \[ y' = 6x + 6 \]

- **Constant multiple rule** If \( c \) is a constant and \( f \) is a differentiable function, then
  \[
  \frac{d}{dx} (cf(x)) = c \frac{d}{dx} f(x).
  \]

- **Sum rule** If \( f \) and \( g \) are both differentiable functions, then
  \[
  \frac{d}{dx} (f(x) + g(x)) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x).
  \]
  This rule can also be written as \((f + g)' = f' + g'\).

- **Product rule** If \( f \) and \( g \) are both differentiable functions, then
  \[
  \frac{d}{dx} (f(x)g(x)) = f(x) \frac{d}{dx} g(x) + g(x) \frac{d}{dx} f(x).
  \]
  This rule can also be written as \((fg)' = fg' + gf'\).

- **Quotient rule** If \( f \) and \( g \) are both differentiable functions, then
  \[
  \frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{g(x) \frac{d}{dx} f(x) - f(x) \frac{d}{dx} g(x)}{(g(x))^2}.
  \]
  This rule can also be written as \( \left( \frac{f}{g} \right)' = \frac{gf' - fg'}{g^2} \).
Example 3.3.1 Determine the derivative of each of the following functions.

a) \( f(x) = x^3 + 4x - 1 \)
\[
f'(x) = 3x^2 + 4 \]

b) \( g(u) = \sqrt{u}(2u - u^2) \)
\[
g'(u) = \frac{3}{2}u^{1/2} - \frac{5}{2}u^{3/2}
\]
\[
= 3\sqrt{u} - \frac{5\sqrt{u^3}}{2}
\]

let \( \frac{\sqrt{u}}{u} = \frac{u^{1/2}}{u} = 2u - u^2 \)
\[
f' = \frac{1}{2}u^{-1/2} - \frac{1}{2}u^{3/2}
\]
\[
g' = \frac{u^{1/2}(2-2u)}{2} - \frac{1}{2}u^{3/2}
\]
\[
= 2\sqrt{u} - 2u^{1/2} + \frac{u^{1/2}}{2}
\]
\[
= \frac{3\sqrt{u} - \frac{5}{2}u^{3/2}}{2}
\]

MATH1050, 2011. Section 3. Page 76
\[ = \frac{3n^6 - 36n^4 - 24n^3}{9n^8} \]

\[ = \frac{1}{3n^3} \left( n^3 - 12n - 8 \right) \]

\[ = \frac{39n^8}{3n^8} \cdot \frac{n^3 - 12n - 8}{3n^5} \]

\[ a^{-m} = \frac{1}{a^m} \]

\[ h(n) = \left( \frac{4n-n^3+2}{3n^4} \right) \left( 3n^4 \right)^{-1} \]

\[ u = 3n^4, \quad y = u^{-1} \]

\[ u' = 12n^3, \quad y' = -u^{-2} \]

\[ g^{-1}(n) = -12n^3 \cdot u^2 \]

\[ = -12n^3 \cdot (3n^4)^{-2} \]

\[ x = \frac{\frac{\partial}{\partial \rho} \left( h \cdot \frac{x}{\rho} \right)}{x} = \frac{x}{\rho} \cdot \frac{\partial}{\partial \rho} \left( h \cdot \frac{x}{\rho} \right) \]

\[ x = \frac{2}{h} \]
• **Chain rule** (also called the **Composite Function rule**). If \( f \) and \( g \) are both differentiable functions, then

\[
(f \circ g)'(x) = f'(g(x))g'(x).
\]

If \( y = f(u) \) and \( u = g(x) \), then this rule can be written as

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.
\]

• To apply the chain rule, think about starting with the outside function and working your way in.

---

**Example 3.3.2**  a) Find \( h'(x) \) where \( h(x) = (2x^3 - 3x + 1)^4 \).

\( y = x^4 \)

\( y' = 4x^3 \)

Let \( u = 2x^3 - 3x + 1 \)

\( h(u) = u^4 \)

\( u' = 6x^2 - 3 \)

\( h'(u) = 4u^3 \)

\[
h'(x) = (6x^2 - 3) \cdot 4u^3
\]

\[
= 4(6x^2 - 3)(2x^3 - 3x + 1)^3
\]

---

continued...
Example 3.3.2 (continued)

b) Find $h'(x)$ where $h(x) = \sqrt{x^2 - 3x} = \left(\frac{x^2 - 3x}{u}\right)^{1/2}$

Let $u = x^2 - 3x$ then $h = \frac{1}{2} u^{-1/2}$

$u' = 2x - 3$ \hspace{1cm} $h'(u) = \frac{1}{2} u^{-1/2}$

$\therefore \quad h'(x) = \frac{(2x-3)}{2} u^{-1/2}$

$= \frac{2x-3}{2 \sqrt{x^2-3x}}$


c) Find $h'(x)$ where $h(x) = \frac{1}{(-10x^{-7} + 4)^2}$

$= \left(\frac{-10x^{-7} + 4}{\text{outer deriv.}}\right)^{-2}$

$h'(x) = -2 \left(-10x^{-7} + 4\right)^{-3} \times \frac{70x^{-8}}{\text{inner deriv.}}$

$= \frac{-140}{x^9(-10x^{-7} + 4)^3}$
• So far we have only looked at deriving relations where $y$ or $f(x)$ is defined explicitly as a relation of $x$. What happens if $y$ is defined implicitly as a relation of $x$, such as $y^2 = x$ or $y^3 + 2y = 4x^3$? We can use a technique called implicit differentiation, which is based on the chain rule.

\[ y^2 = \frac{25 - x^2}{y} \]

• Let’s look at the derivative of $x^2 + y^2 = 25$ (circle, radius 5, centred at the origin). We differentiate both sides w.r.t $x$,

\[ \frac{d}{dx} (x^2 + y^2) = \frac{d}{dx} (25). \]

\[ \Rightarrow \frac{d}{dx} (x^2) + \frac{d}{dx} (y^2) = 0. \]

We can easily work out $\frac{d}{dx} (x^2)$, but how can we do $\frac{d}{dx} (y^2)$?

This is where the chain rule comes in.

\[ \frac{d}{dx} (y^2) = \frac{d}{dy} (y^2) \frac{dy}{dx} = 2y \frac{dy}{dx} \]

So we now have

\[ 2x + 2y \frac{dy}{dx} = 0 \]

\[ \Rightarrow \frac{dy}{dx} = -\frac{2x}{2y} \]

\[ \Rightarrow \frac{dy}{dx} = -\frac{x}{y} \]

So the derivative of $y$ w.r.t $x$, where $y$ is defined in terms of $x$ implicitly by the equation $x^2 + y^2 = 25$, is $\frac{dy}{dx} = -\frac{x}{y}$. 

\[ y = x^2 \]

\[ y' = 2x \]

\[ \frac{dy}{dx} = 2x \]
Example 3.3.3  Determine the derivative of each of the following functions.

a) \( y^3 = x^3 + 4x - 1 \)

\[
3y^2 \frac{dy}{dx} = 3x^2 + 4
\]

\[
\frac{dy}{dx} = \frac{3x^2 + 4}{3y^2}
\]

b) \( \sqrt{y} = \sqrt{2u - u^2} \)

\[
y = u(2u - u^2)^2
\]

\[
\frac{dy}{du} = \frac{u(2u - u^2)^2}{2(2u - u^2)}
\]

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}
\]

\[
\frac{1}{2} \sqrt{y} \frac{dy}{dx} = 3u \sqrt{2u - u^2} - \frac{u}{2} \frac{d}{dx} (2u - u^2)
\]

\[
\frac{dy}{dx} = \frac{3u \sqrt{2u - u^2} - \frac{u}{2} (2 - 2u)}{\frac{1}{2} \sqrt{y}}
\]

\[
= 2y \sqrt{y} \left( \frac{3u \sqrt{2u - u^2} - \frac{u}{2} (2 - 2u)}{3u - \frac{5}{2}u^2} \right)
\]

c) \( y^3 + 2y + 7 = x^3 + 4x - 1 \)

\[
\frac{d}{dx} \left( y^3 + 2y + 7 \right) = \frac{d}{dx} \left( x^3 + 4x - 1 \right)
\]

\[
(3y^2 + 2) \frac{dy}{dx} = 3x^2 + 4
\]

\[
\frac{dy}{dx} = \frac{3x^2 + 4}{3y^2 + 2}
\]

\[
\frac{d}{dx} \left( y^3 + 2y + 7 \right) = \frac{d}{dy} \left( y^3 + 2y + 7 \right) \frac{dy}{dx}
\]

\[
= \frac{d}{dy} \left( y^3 + 2y + 7 \right) \frac{dy}{dx}
\]

\[
= \frac{3y^2 + 2}{3y^2 + 2}
\]
\[ y' = \frac{1}{3} (x^3 + 4x + 1)^{-2/3} \times (3x^2 + 4) \]

\[ y' = \frac{3x^2 + 4}{3(x^3 + 4x + 1)^{2/3}} \]

Is \( y^2 = \frac{x^3 + 4x + 1}{(x^3 + 4x + 1)^{2/3}} \)?

We know \( y^3 = x^3 + 4x + 1 \).

\[ (y^3)^x = y^x \]

\[ x = \frac{2}{3} \]
Derivatives of trigonometric functions

- Note that when we use the trigonometric functions, such as $f(x) = \sin x$, all angles are measured in radians.
- To determine the derivative of $f(x) = \sin x$ we require two special limits:
  \[
  \lim_{h \to 0} \frac{\sin h}{h} = 1, \quad \lim_{h \to 0} \frac{\cos h - 1}{h} = 0.
  \]
  The proofs of these limits can be found in most calculus textbooks, but we won’t worry about the proofs in this course.

- If $f(x) = \sin x$ then $f'(x) = \cos x$.

**Proof**

\[
\begin{align*}
  f'(x) &= \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h} \\
        &= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
        &= \lim_{h \to 0} \left( \frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right) \\
        &= \lim_{h \to 0} \sin x \left( \frac{\cos h - 1}{h} \right) + \cos x \left( \frac{\sin h}{h} \right) \\
        &= \sin x \cdot 0 + \cos x \cdot 1 \\
        &= \cos x.
\end{align*}
\]
• Using a similar method to the previous slide, we can show that if $f(x) = \cos x$ then $f'(x) = -\sin x$.

**Example 3.3.4** Use the quotient rule to determine $\frac{d}{dx} \tan x$.

$y = \tan x = \frac{\sin x}{\cos x}$

Let $u = \sin x$, $v = \cos x$.

$u' = \cos x$, $v' = -\sin x$.

$y = \frac{u'v - uv'}{v^2}$

$= \frac{(\cos x \cdot \cos x) - (\sin x \cdot \sin x)}{\cos^2 x}$

$= \frac{\cos^2 x}{\cos^2 x}$

$= \frac{1}{\cos^2 x} = \sec^2 x$.

• The derivatives of the trigonometric functions are summarised below.

\[
\begin{align*}
\frac{d}{dx} (\sin x) &= \cos x \\
\frac{d}{dx} (\cos x) &= -\sin x \\
\frac{d}{dx} (\tan x) &= \sec^2 x \\
\frac{d}{dx} (\csc x) &= -\csc x \cot x \\
\frac{d}{dx} (\sec x) &= \sec x \tan x \\
\frac{d}{dx} (\cot x) &= -\csc^2 x
\end{align*}
\]
Derivatives of exponential and logarithmic functions

- If \( f(x) = e^x \), then \( f'(x) = e^x \).
- Thus, on the graph of \( f(x) = e^x \), the slope of the tangent line at each point is equal to the function value at that point.
- If \( f(x) = \ln x \), then \( f'(x) = \frac{1}{x} \). Note that here \( x > 0 \) since the domain of \( f(x) = \ln x \) is \( x > 0 \).

**Example 3.3.5** Determine the derivative of each of the following functions.

**a)** \( f(x) = x^2 e^{2x} \)

\[
\begin{align*}
    u &= x^2, \\
    v &= e^{2x}, \\
    u' &= 2x, \\
    v' &= 2e^{2x}, \\
    f'(x) &= uv' + vu' \\
    &= (x^2)(2e^{2x}) + (2x)(e^{2x}) \\
    &= 2xe^{2x} + 2xe^{2x} \\
    &= 2x^2e^{2x} + e^{2x} \\
    &= e^{2x}(2x^2 + 1).
\end{align*}
\]

**b)** \( g(x) = \ln(x^3 - 2x) \)

Let \( u = x^3 - 2x \)

\[
\begin{align*}
    \frac{du}{dx} &= 3x^2 - 2, \\
    \frac{dg}{du} &= \frac{1}{u} \\
    \frac{dg}{dx} &= \frac{d}{dx} \left( \ln u \right) \\
    &= \left(3x^2 - 2\right) \frac{1}{u} \\
    &= \frac{3x^2 - 2}{x^3 - 2x}.
\end{align*}
\]
3.4 Critical points and curve sketching

- When solving practical problems, or when sketching the graph of a function, we often need to know when a function attains a maximum or minimum value.

- A function $f$ has a **global maximum** at $c$ if $f(c) \geq f(x)$ for all $x$ in the domain of $f$. The number $f(c)$ is called the maximum value of $f$ on its domain. A global maximum is also called an **absolute maximum**.

- A function $f$ has a **global minimum** at $c$ if $f(c) \leq f(x)$ for all $x$ in the domain of $f$. The number $f(c)$ is called the minimum value of $f$ on its domain. A global minimum is also called an **absolute minimum**.

- A function $f$ has a **local maximum** at $c$ if $f(c) \geq f(x)$ for all $x$ near $c$.

- A function $f$ has a **local minimum** at $c$ if $f(c) \leq f(x)$ for all $x$ near $c$.

**Example 3.4.1** On the graph of the function $f$ below, identify all global and local maxima and minima.
• If \( f \) has a local maximum or minimum at \( a \), and if \( f'(a) \) exists, then \( f'(a) = 0 \).

• The point \( (a, f(a)) \) is a critical point of the function \( f \) if \( f'(a) = 0 \) or if \( f'(a) \) does not exist (but \( f(a) \) does).

• Thus, all local maxima and minima are critical points. Note however, that not all critical points are local maxima or minima.

• To find any local maxima or minima of a function \( f \), we solve the equation \( f' = 0 \). We can then classify any critical points we find using the information about the function near the critical point.

• A function \( f \) is strictly increasing on an interval \( [a, b] \) if for all \( x_1 \) and \( x_2 \) in \( [a, b] \), \( f(x_1) < f(x_2) \) whenever \( x_1 < x_2 \).

• A function \( f \) is strictly decreasing on an interval \( [a, b] \) if for all \( x_1 \) and \( x_2 \) in \( [a, b] \), \( f(x_1) > f(x_2) \) whenever \( x_1 < x_2 \).

• If \( f'(x) > 0 \) on an interval, then \( f \) is strictly increasing on that interval.

• If \( f'(x) < 0 \) on an interval, then \( f \) is strictly decreasing on that interval.

• If \( f'(x) = 0 \) on an interval, then \( f \) is constant on that interval.
• **First derivative test** Suppose that the function $f$ has a critical point at $x = c$. Then

- If $f'$ changes sign from positive to negative at $c$, then $f$ has a local maximum at $c$.
- If $f'$ changes sign from negative to positive at $c$, then $f$ has a local minimum at $c$.
- If $f'$ does not change sign at $c$, then $f$ has neither a local maximum nor a local minimum at $c$.

**Example 3.4.2** Find all local maxima and minima of $f(x) = x^3 - 2x^2 + x + 1$ and classify them using the first derivative test. Use this information to sketch the graph of $f$.

$y' = f'(x) = 3x^2 - 4x + 1$

Crit. points when $f'(x) = 0$

$= 3x^2 - 4x + 1$

$= (3x - 1)(x - 1)$

$3x - 1 = 0 \quad \text{or} \quad x - 1 = 0$

$x = \frac{1}{3}, \frac{1}{27}$

$y = \frac{1}{27} - \frac{2}{9} + \frac{1}{3} \quad y = \frac{31}{27}$

$f'(0) = +ve$

$f'(\frac{1}{3}) = -ve$

$f'(\frac{1}{2}) = +ve$

$f'(1) = -ve$

$f'(\frac{1}{3}, \frac{1}{27}) \text{ is a local max}$

$f'(\frac{1}{2}) \text{ is a local min}$

$\text{Graph of } f(x)$
The second derivative

- The second derivative of a function $f$ is the derivative of the derived function $f'$. The second derivative of $f$ is denoted $f''$. \[ \frac{d^2 y}{dx^2} \]

- The second derivative provides information about the concavity of the graph of a function.

- If the graph of $f$ lies above all of its tangent lines on an interval, then it is \textit{concave up} on that interval. If the graph of $f$ lies below all of its tangent lines on an interval, then it is \textit{concave down} on that interval.

- If $f''(x) > 0$ for all $x$ in an interval, then the graph of $f$ is \textit{concave up} on that interval. If $f''(x) < 0$ for all $x$ in an interval, then the graph of $f$ is \textit{concave down} on that interval.

- **Second derivative test** Suppose $f''$ is a continuous function near a point $c$.

  \[ \begin{align*}
  &\cup \quad \text{If } f'(c) = 0 \text{ and } f''(c) > 0, \text{ then } f \text{ has a local minimum at } c. \\
  &\cap \quad \text{If } f'(c) = 0 \text{ and } f''(c) < 0, \text{ then } f \text{ has a local maximum at } c.
  \\
  &\quad \text{If } f''(c) = 0, \text{ then this test is inconclusive.}
  \end{align*} \]
**Example 3.4.3** Analyse the curve \( y = x^4 - 2x^3 \) with respect to concavity, and local maxima and minima. Use this information to sketch the curve.

\[
y = x^4 - 2x^3
\]
\[
y' = 4x^3 - 6x^2
\]
Critical when \( y' = 0 \)
\[
0 = 4x^3 - 6x^2
\]
\[
x^2(2x - 3) = 0
\]
\[
x = 0 \quad \text{or} \quad 2x - 3 = 0
\]
\[
x = 0 \quad \text{or} \quad x = \frac{3}{2}
\]
\[
y = 0
\]
\[
y''(0) = 12x^2 - 12x
\]
\[
y''(0) = 0 \quad ??
\]
\[
y''\left(\frac{3}{2}\right) = +\text{ve} \quad \Rightarrow \text{local min.}
\]
1st derivative test for \( x = 0 \)
\[
y'(-1) = -4 - 6 = -10 \quad -\text{ve}
\]
\[
y'(1) = 4 - 6 = -2 \quad -\text{ve}
\]
\[
\therefore (0, 0) \text{ is neither a local min. or max; it is a point of horizontal inflection.}
\]

MATH1050, 2011. Section 3.  Page 88
**Curve sketching** To sketch the curve of a function \( y = f(x) \):

- Determine the **domain of** \( f \).
- Determine the **y-intercept of** the graph by evaluating \( f(0) \).
- If it is possible to solve the equation \( f(x) = 0 \), find the **x-intercepts of** the graph.
- Determine \( f' \) and identify the intervals on which \( f \) is increasing and the intervals on which \( f \) is decreasing.
- Find the **critical points of** \( f \). Determine which critical points are local maxima or local minima (use first or second derivative test).
- Determine \( f'' \) and identify the intervals on which \( f \) is concave up and the intervals on which \( f \) is concave down.
- Sketch the graph.

**Example 3.4.4** Sketch the graph of the function \( f(x) = x^2 e^x \).

\[
y = \frac{d}{dx}(x^2 e^x) \quad \text{product rule}
\]

let \( u = x^2 \), \( v = e^x \)

\[
u' = 2x, \quad v' = e^x
\]

\[
y' = x^2 e^x + 2xe^x
\]

crit. pts. when \( y' = 0 \)

\[
0 = xe^x + 2xe^x
\]

\[
0 = e^x(x^2 + 2x)
\]

\[
= xe^x(x + 2)
\]

continued...
Example 3.4.4 (continued) Extra space for your work.

\[ 0 = e^x (x^2 + 2x) \]

\[ e^x = 0 \quad \Rightarrow \quad x^2 + 2x = 0 \]

\[ x(x+2) = 0 \]

\[ x = 0 \quad \text{or} \quad x+2 = 0 \]

\[ x = -2 \]

\[ y = 4e^{-2} = \frac{4}{e^2} \]

\[ (-2, \frac{4}{e^2}) \]

\[ y' = xe^x + 2xe^x \]

\[ y'' = xe^x + 2xe^x + 2xe^x + 2e^x \]

\[ y''(0) = 2 + \text{ve} \quad \text{local min} \quad U \]

\[ y''(-2) = \frac{4}{e^2} - \frac{4}{e^2} - \frac{4}{e^2} + \frac{4}{e^2} = -\text{ve} \quad \text{local max} \]
3.5 Applications of differentiation

Optimisation problems

**Example 3.5.1** Find two numbers whose difference is 100 and whose product is a minimum.

Let \( x \) be the first number, \( y \) the second.

\[
\begin{align*}
  x - y &= 100 \\
  x &= 100 + y \\
  y &= x - 100 \\
  (x, y) &= (100, 0)
\end{align*}
\]

Want to minimise \( P = xy \)

\[
\begin{array}{|c|c|c|}
\hline
x & y & P \\
\hline
0 & 100 & 0 \\
10 & 110 & 1010 \\
50 & 150 & 7500 \\
-10 & 90 & -900 \\
-100 & 0 & 0 \\
-110 & 10 & -1100 \\
-200 & -100 & +P \\
\hline
\end{array}
\]

\[
\begin{align*}
  P(x) &= x(100 + x) \\
  P(y) &= y(100 + y) \\
  P(x) &= x^2 + 100x \\
  P'(x) &= 2x + 100 \\
  &\text{when } P'(x) = 0 \text{ } \Rightarrow \text{ } 0 = 2x + 100 \\
  &\Rightarrow x = -50 \\
  y &= -50 \\
  P(x) &= 50 \times -50 \\
  &= -2500
\end{align*}
\]

\[
P''(x) = 2 \quad \text{positive} \text{ : local min. at}
\]

\[
x = 50, \quad y = -50.
\]
Example 3.5.2: A box with a square base and open top must have a volume of exactly 4000 cm$^3$, a height of at least 5 cm, and a base side length of at least 5 cm.

a) Find the dimensions of the box that minimise the amount of material used.

\[ V = l \times w \times h \]

\[ V = b^2 \cdot h \] (square base)

\[ h = \frac{4000}{b^2} \]

\[ S.A. = b^2 + 4bh \]

\[ S.A. (b) = b^2 + 4b \cdot \frac{4000}{b^2} \]

\[ S.A. = b^2 + \frac{16000}{b} \]

\[ S' (b) = 2b - \frac{16000}{b^2} = 0 \]

Critical points when \( S' (b) = 0 \)

\( 2b = \frac{16000}{b^2} \)

\( b^3 = 8000 \)

\( b = \sqrt[3]{8000} = 20 \text{ cm} \)

When \( b = 20, \ h = \frac{4000}{400} = 10 \text{ cm} \)

Dim. are \( 20 \times 20 \times 10 \text{ cm} \)

continued...
Example 3.5.2 (continued)  

b) Find the dimensions of the box that maximise the amount of material used.

There was a local min. but no local max. So I need to find the global max.

\[ h = \frac{40000}{b^2} \quad \text{(from a1)} \]

\[ b^2 = \frac{40000}{h} \]

When \( h = 5 \),
\[ b^2 = \frac{40000}{5} = 8000 \]
\[ b = \sqrt{8000} = 10\sqrt{8} \]
\[ = 10 \times 2\sqrt{2} \]
\[ = 20\sqrt{2} \]

Check SA when \( b = 5 \) and \( h = 5 \).

\[ S(5) = b^2 + \frac{16000}{b} \]
\[ S(5) = 25 + \frac{16000}{5} = 3225 \text{ cm}^2 \]

\[ S(20\sqrt{2}) = (20\sqrt{2})^2 + \frac{16000}{20\sqrt{2}} \]
\[ = 800 + \frac{800}{\sqrt{2}} \]
\[ \approx 1365.7 \text{ cm}^2 \]

\[ S.A. \text{ is max. when } b = 5 \text{ cm} \]
\[ 5 \times 5 \times 160 \text{ cm}^3 \]

\[ 5 \times 5 \times 160 \text{ cm}^3 \]
Rates of change $= \text{DERIV.} = \text{SCOPE}$.

- Rates of change have important applications in many areas. The derivative of a function $f$ with respect to a variable $x$ gives the rate of change of $f$ with respect to $x$.

- If $s(t)$ is a function of displacement with respect to time, then the derivative $s'(t)$ gives the velocity, since velocity is the rate of change of displacement with respect to time. Similarly, $s''(t)$ gives the acceleration, since acceleration is the rate of change of velocity with respect to time.

**Example 3.5.3** A skier pushes off and heads directly down a ski slope with an initial velocity of 3 m/s. The distance the skier has travelled after $t$ seconds is given by $s(t) = 3t + 2t^2$.

**a)** Determine the velocity of the skier after 2 seconds.

$v(t) = s'(t) = 3 + 4t$

$\therefore v(2) \approx 11 \text{ m/s}$.

**b)** When the skier reaches a velocity of 15 m/s (that is 54 km/h), he makes a turn to slow down. How far has the skier travelled down the hill when he starts the turn?

$v(t) = 15$

$15 = 3 + 4t$

$\Rightarrow t = 3 \text{ sec}$

$s(3) = 3 \times 3 + 2 \times 3^2$

$= 27 \text{ m}$
• In economics, it is often important to study the functions that describe the cost of producing various quantities of a given product.
• Let $C(x)$ be the total cost to produce $x$ units of a certain commodity. The **marginal cost** is the rate of change of the cost function with respect to the number of units produced.
• It can be shown that the marginal cost is approximately equal to the cost of producing one more unit of product.

$$C'(x) \approx \frac{C(x+1) - C(x)}{1}$$

**Example 3.5.4** A company has estimated that the cost (in dollars) of producing $x$ snowboards is

$$C(x) = 10000 + 18x + 0.02x^2.$$  

a) Determine the marginal cost function and evaluate $C'(10)$.

$$C'(x) = 18 + 0.04x$$

$$C'(10) = 18 + 0.04 \times 10 = 18.40$$

b) Compare $C'(10)$ with the actual cost of producing the 11th snowboard.

$$\frac{C(11) - C(10)}{11 - 10} = \frac{10000 + 18 \times 11 + 0.02 \times (11^2) - (10000 + 18 \times 10 + 0.02 \times 10^2)}{11 - 10}$$

$$= 18 + 0.04x + 0.02$$

When $x = 10$, $A.C = 18.02 + 0.04 \times 10 = 18.42$.  

MATH1050, 2011. Section 3.  
Page 95