8 Sequences and Series

We are dealing with two related concepts in this chapter, sequences and series. A sequence is an ordered set of numbers, and these numbers are called the terms of the sequence. A series is the sum of all of the terms of a sequence. We will see later the use of these in mathematical proofs and applications to population growth.

8.1 Introduction to Sequences

There are two types of sequences based on the number of terms; a finite sequence is one with a fixed number of terms, while an infinite sequence is one with an infinite number of terms. A typical representation of a sequence is \( b_1, b_2, b_3, \ldots, b_6 \) where \( b_1 \) is the first term and \( b_6 \) is the last term. The sequence as a whole can be written as \( \{b_n\} \) or \( \{b_n\}_{n=1}^6 \). Note that the sequence does not have to have first term of the form \( b_1 \). For example, the sequence \( c_3, c_4, \ldots, c_{10} \) can be written as \( \{c_n\}_{n=3}^{10} \). For an infinite sequence we can write \( \{c_n\}_{n=1}^{\infty} \).

Examples List the first five terms of the following sequences.

- \( \{a_n\}_{n=1}^{10} \), with \( a_n = n^2 + 1. \)
  \[
  \begin{align*}
  a_1 &= 1^2 + 1 = 2 \\
  a_2 &= 2^2 + 1 = 5 \\
  a_3 &= 3^2 + 1 = 10 \\
  a_4 &= 4^2 + 1 = 17 \\
  a_5 &= 5^2 + 1 = 26 
  \end{align*}
  \]

- \( \{b_n\}_{n=0}^{\infty} \), with \( b_n = 100 - 3n. \)
  Notice that this sequence starts with \( n = 0 \) rather than \( n = 1. \)
  \[
  \begin{align*}
  b_0 &= 100 - 3 \times 0 = 100 \\
  b_1 &= 100 - 3 \times 1 = 97 \\
  b_2 &= 100 - 3 \times 2 = 94 \\
  b_3 &= 100 - 3 \times 3 = 91 \\
  b_4 &= 100 - 3 \times 4 = 88 
  \end{align*}
  \]
• \( \{c_n\}_{n=1}^5 \), with \( c_n = \sin \left( \frac{6\pi}{n} \right) \).

\[
\begin{align*}
c_1 &= \sin(6\pi) = 0 \\
c_2 &= \sin(3\pi) = 0 \\
c_3 &= \sin(2\pi) = 0 \\
c_4 &= \sin \left( \frac{3\pi}{2} \right) = -1 \\
c_5 &= \sin \left( \frac{6\pi}{5} \right) \approx -0.588
\end{align*}
\]

• \( \{d_n\}_{n=1}^\infty \), with \( d_n = n^3 - 6n^2 + 11n - 6 \).

\[
\begin{align*}
d_1 &= 1^3 - 6 \times 1^2 + 11 \times 1 - 6 = 0 \\
d_2 &= 2^3 - 6 \times 2^2 + 11 \times 2 - 6 = 0 \\
d_3 &= 3^3 - 6 \times 3^2 + 11 \times 3 - 6 = 0 \\
d_4 &= 4^3 - 6 \times 4^2 + 11 \times 4 - 6 = 6 \\
d_5 &= 5^3 - 6 \times 5^2 + 11 \times 5 - 6 = 60
\end{align*}
\]

Notice that the first three terms of \( \{c_n\}_{n=1}^5 \) and \( \{d_n\}_{n=1}^\infty \) but the sequences are not the same. Specifying the first few terms of a sequence is not sufficient to uniquely determine the sequence.

In these examples we specified the sequence by what is called the *closed form*, a rule of the former \( b_n = f(n) \) where \( f \) is a function of \( n \). Alternatively we can give a recursive definition which has initial conditions and relies on using the value of one or more of the previous terms to find the next one. For example, the sequence 1, 1, 2, 3, 5 can be defined recursively as \( b_n = b_{n-1} + b_{n-2} \) \( (n = 2, 3, 4) \) with \( b_0 = 1 \) and \( b_1 = 1 \).

**Examples** Find a recursive definition and a closed form for each of the following sequences.

• 4, 7, 10, 13, 16

To get from one term to the next we add 3, so a recursive definition is \( a_n = a_{n-1} + 3 \) \( (n = 2, 3, 4, 5) \), \( a_1 = 4 \).

One possible closed form is \( \{a_n\}_{n=1}^5 \), with \( a_n = 3n + 1 \), which can also be written as \( \{3n + 1\}_{n=1}^5 \).

• 3, 9, 27, 81

To get from one term to the next we multiply by 3, so a recursive definition is \( a_n = 3a_{n-1} \) \( (n = 2, 3, 4) \), \( a_1 = 3 \).

One closed form is \( \{3^n\}_{n=1}^4 \).
For a finite sequence it is possible to describe the sequence by writing down all of the terms, by giving a closed form, or by giving a recursive definition including initial conditions. For an infinite sequence it is not possible to list all of the terms so the sequence must be described by giving a closed form, or by giving a recursive definition including initial conditions.

8.2 Arithmetic and Geometric Sequences

Arithmetic Sequences

An arithmetic sequence (A.S.) is a sequence in which the difference between any two successive terms is a constant, called the common difference. An arithmetic sequence \( \{a_n\}_{n=1}^m \), for a finite sequence, or \( \{a_n\}_{n=1}^\infty \), for an infinite sequence, with first term \( a \) and common difference \( d \) has a closed form \( a_n = a + (n - 1)d \).

Examples

1. Determine a closed form for the following arithmetic sequences.

   - \(-4, 1, 6, 11, 16\)
     
     We can see that the first term is \(-4\) so \(a = -4\). The common difference is \(d = 5\). Therefore,
     
     \[
     a_n = -4 + (n - 1) \times 5 \\
     = -4 + 5n - 5 \\
     = -9 + 5n \\
     = 5n - 9
     \]

   - \(10\frac{1}{2}, 9\frac{3}{4}, 9, 8\frac{1}{4}\)
     
     This arithmetic sequence has \(a = 10\frac{1}{2}\) and \(d = -\frac{3}{4}\).
     
     \[
     a_n = 10\frac{1}{2} + (n - 1) \times \left(\frac{3}{4}\right) \\
     = 10\frac{1}{2} - \frac{3n}{4} + \frac{3}{4} \\
     = 11\frac{1}{4} - \frac{3n}{4}
     \]

2. The first 3 terms of an arithmetic sequence are 100, 93, 86. Which will be the first negative term?
First we determine the closed form of the sequence. We can see that \( a = 100 \) and \( d = -7 \) so

\[
a_n = 100 + (n - 1) \times (-7) \\
= 100 - 7n + 7 \\
= 107 - 7n.
\]

To find when this will be negative we solve \( a_n < 0 \).

\[
a_n < 0 \\
107 - 7n < 0 \\
107 < 7n \\
\frac{107}{7} < n \\
n > 15.29
\]

Therefore the sixteenth term is the first term which will be negative, \( a_{16} = 107 - 7 \times 16 = -5 \).

Geometric Sequences

A geometric sequence (G.S) is a sequence in which every term is equal to the previous term multiplied by a particular number, so there is a common ratio between any two successive terms. A geometric sequence \( \{a_n\}_{n=1}^m \), for a finite sequence, or \( \{a_n\}_{n=1}^\infty \), for an infinite sequence, with first term \( a \) and common ratio \( r \) has a closed form \( a_n = ar^{n-1} \).

Examples

1. Determine which of the following is a geometric sequence and find a closed form for any which are a geometric sequence.

   - 6, 15, 37.5, 93.75
     
     We can find the common ratio by dividing any term by the one preceding it.
     
     \[
     \frac{15}{6} = \frac{37.5}{15} = \frac{93.75}{37.5} = 2.5
     \]
     
     This is a geometric sequence with \( r = 2.5 \) and \( a = 6 \), so, \( a_n = 6 \times 2.5^{n-1} \).

   - 4, 8, 24, 48
     
     This is not a geometric sequence since \( \frac{8}{4} = \frac{48}{24} = 2 \) but \( \frac{24}{8} = 3 \).

   - 99, -33, 11, -\( \frac{11}{3} \)
     
     This sequence has common ratio
     
     \[
r = \frac{-33}{99} = \frac{11}{-33} = \frac{-\frac{11}{3}}{\frac{11}{3}} = -\frac{1}{3}
     \]
The first term is \( a = 99 \) so a closed form is \( a_n = 99 \times \left( \frac{-1}{3} \right)^{n-1} \).

2. The third term of a geometric sequence is 96 and the sixth term is 6144. Determine a closed form for the sequence.

We know that \( a_3 = 96 \) and \( a_6 = 6144 \) so we use the formula \( a_n = ar^{n-1} \) to create two equations.

\[

da_3 &= ar^{3-1} \\
&= ar^2 \\
&= 96 \\

da_6 &= ar^{6-1} \\
&= ar^5 \\
&= 6144
\]

We can use the two equations \( ar^2 = 96 \) and \( ar^5 = 6144 \) to find \( a \) and \( r \). Rearranging the first of these we have \( a = \frac{96}{r^2} \) which we can substitute into \( ar^5 = 6144 \) to find \( r \).

\[
\frac{96}{r^2} \times r^5 = 6144 \\
96r^3 = 6144 \\
r^3 = 64 \\
r = 4
\]

We can now substitute this value of \( r \) into \( ar^2 = 96 \) to find \( a \).

\[
\begin{align*}
a \times 4^2 &= 96 \\
16a &= 96 \\
a &= 6
\end{align*}
\]

So a closed form for this geometric sequence is \( a_n = 6 \times 4^{n-1} \).

### 8.3 Arithmetic and Geometric Series

We are also often interested in the sum of the terms of a sequence. We can express the sum of the sequence \( \{a_n\}_{n=1}^6 \) with \( a_n = n^3 \) as

\[
\sum_{i=1}^{6} i^3.
\]
The letter \( i \) in this case is just a dummy variable and another could be used. For example, the above sum could be expressed as

\[
\sum_{k=1}^{6} k^3.
\]

This is called sigma notation or sum notation.

**Examples**

1. *Calculate the value of the following sums.*

   - \( \sum_{i=1}^{6} i^3 \)
     
     \[
     \sum_{i=1}^{6} i^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 = 441
     \]

   - \( \sum_{i=7}^{8} (3i + 1) \)
     
     \[
     \sum_{i=7}^{8} (3i + 1) = 3 \times 7 + 1 + 3 \times 8 + 1 = 47
     \]

   - \( \sum_{i=10}^{10} i \)
     
     \[
     \sum_{i=10}^{10} i = 10
     \]
2. Write each of the following in sigma notation.

- $2 - 4 + 8 - 16 + 32 - 64$
  We can see that each term is of the form $2^i$ and that each of the even numbered terms is negative. We use $(-1)^{i+1}$ to make these negative values for each even numbered term. There are six terms, so combining all of this information in sigma notation we have

$$\sum_{i=1}^{6} (-1)^{i+1}2^i.$$  

- $5x^2 + 10x^3 + 15x^4 + 20x^5 + 25x^6$
  We can see that each term has a multiple of 5 and a power of $x$. For each term the coefficient can be written as $5i$ and the $x$ power as $x^{i+1}$. So, in sigma notation we have

$$\sum_{i=1}^{5} 5ix^{i+1}.$$  

A series is the sum of the terms of a sequence. For a sequence $\{a_n\}_{n=1}^{N}$, the corresponding series is

$$\sum_{n=1}^{N} a_n.$$  

We use the notation $S_n$ to denote the sum of the first $n$ terms of a sequence. For an arithmetic sequence with first term $a$ and common difference $d$,

$$S_n = \frac{n}{2}(2a + (n-1)d).$$

For a geometric sequence with first term $a$ and common ratio $r$,

$$S_n = \frac{a(1 - r^n)}{1 - r}.$$

Examples

- An arithmetic sequence has first terms 2, 7, 12, 17. If the corresponding arithmetic series is equal to 245 calculate the number of terms in the sequence.
  We can see that $a = 2$ and $d = 5$ so we substitute these into the formula for
the sum of the sequence.

\[ S_n = \frac{n}{2}(2a + (n - 1)d) \]
\[ = \frac{n}{2}(2 \times 2 + (n - 1) \times 5) \]
\[ = \frac{n}{2}(4 + 5n - 5) \]
\[ = \frac{n}{2}(5n - 1) \]
\[ = \frac{5n^2}{2} - \frac{n}{2} \]

Next we need to solve \( S_n = 245 \).

\[ \frac{5n^2}{2} - \frac{n}{2} = 245 \]
\[ \frac{5n^2}{2} - \frac{n}{2} - 245 = 0 \]

We can use the quadratic formula to solve this equation, with \( a = \frac{5}{2} \), \( b = -\frac{1}{2} \), and \( c = -245 \).

\[
\begin{align*}
    n &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
    &= \frac{\frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 - 4 \times \frac{5}{2} \times -245}}{2 \times \frac{5}{2}} \\
    &= \frac{\frac{1}{2} \pm \sqrt{\frac{1}{4} + 2450}}{5} \\
    &= \frac{\frac{1}{2} \pm 49\frac{1}{2}}{5}, \text{ taking only the positive numerator we get} \\
    &= \frac{50}{5} \\
    &= 10
\end{align*}
\]

Therefore, the arithmetic sequence must have had 10 terms for the sum to be 245.

- A geometric sequence has first terms 360, 180, 90, 45. Calculate the fewest number of terms needed for the corresponding series to exceed 700.

We can see that \( a = 360 \) and \( r = \frac{180}{360} = \frac{90}{180} = \frac{45}{90} = \frac{1}{2} \). We substitute these into our formula for the sum of the first \( n \) terms.

\[
\begin{align*}
    S_n &= \frac{360 \left(1 - \left(\frac{1}{2}\right)^n\right)}{1 - \frac{1}{2}} \\
    &= 720 \left(1 - \left(\frac{1}{2}\right)^n\right)
\end{align*}
\]
We now have to find when \( S_n = 700 \) and take the next term as the number of terms needed to exceed 700.

\[
720 \left( 1 - \left( \frac{1}{2} \right)^n \right) = 700 \\
\left( 1 - \left( \frac{1}{2} \right)^n \right) = \frac{700}{720} \\
\left( 1 - \left( \frac{1}{2} \right)^n \right) = \frac{35}{36} \\
- \left( \frac{1}{2} \right)^n = -\frac{1}{36} \\
\left( \frac{1}{2} \right)^n = \frac{1}{36} \\
n = \frac{\ln \frac{1}{36}}{\ln \frac{1}{2}} \\
n \approx 5.17
\]

We need six terms for the series to exceed 700.

### 8.4 Applications to Population Growth

We can use these techniques for geometric series to model population change using what is called the exponential population model. In this model we only consider births and deaths with a yearly birth rate of \( b\% \) and a yearly death rate of \( d\% \). We let \( P_0 \) be the initial population and \( P_n \) by the population in year \( n \). Then \( P_n = P_0(1 + k)^n \) where \( k = \frac{b-d}{100} \).

**Example** A town is known to have had a population of 32,560 on January 1, 1901. If the birth rate is estimated at 1.5% and the death rate at 1.2%, estimate what the population of the town as on January 1, 1869.

We let 1869 be the date of the initial population \( P_0 \) and so 1901 is \( P_2 \). In this town, \( k = \frac{1.5-1.2}{100} = 0.003 \). So we need to solve \( P_n = P_0(1 + k)^n \) for \( P_0 \).

\[
P_n = P_0(1 + k)^n \\
P_2 = P_0(1 + 0.003)^{32} \\
32,560 = P_0(1.003)^{32} \\
32,560 \approx 1.1006P_0 \\
P_0 \approx 29,583.84
\]

So this town had a population of approximately 29,584 on January 1, 1869.
8.5 Mathematical Induction

We can use a proof technique called *mathematical induction* to prove results about series. The general idea is to prove a statement for a base case and then to prove that if the statement is true for a value \( k \), it must also be true for \( k + 1 \). More formally, for a statement \( P(n) \) about all natural numbers greater than or equal to \( n_0 \) we first have to show that \( P(n_0) \) is true. Then, we show that if \( P(k) \) is true then \( P(k + 1) \) is true for natural numbers \( k \geq n_0 \). In this way we have shown that \( P(n) \) is true for all \( n \geq n_0 \).

Examples

- **Prove that** \( \sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3} \) **for all integers** \( n \geq 1 \).

Let \( P(n) \) be the statement \( \sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3} \). We start with the base case \( P(1) \). The left-hand side (LHS) of \( P(1) \) is \( \sum_{i=1}^{1} i(i+1) = 1 \times 2 = 2 \) and the right-hand side (RHS) of \( P(1) \) is \( \frac{1 \times 2 \times 3}{3} = 2 \) so \( P(1) \) is true. Next, assume that \( P(k) \) is true for some integer \( k \geq 1 \), so \( \sum_{i=1}^{k} i(i+1) = \frac{k(k+1)(k+2)}{3} \). Using this we want to prove that \( P(k+1) \) is true, that is,

\[
\sum_{i=1}^{k+1} i(i+1) = \frac{(k+1)(k+1+1)(k+1+2)}{3} = \frac{(k+1)(k+2)(k+3)}{3}
\]

The LHS of \( P(k+1) \) is

\[
\sum_{i=1}^{k+1} i(i+1) = \sum_{i=1}^{k} i(i+1) + (k+1)(k+1+1)
= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \quad \text{using our inductive assumption}
= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)
= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3}
= \frac{(k+1)(k+2)(k+3)}{3},
\]
which is the RHS of $P(k + 1)$. Therefore, by mathematical induction,
\[ \sum_{i=1}^{n} i(i + 1) = \frac{n(n + 1)(n + 2)}{3}. \]

- **Show that 8 divides $3^{2n} - 1$ for all integers $n \geq 0$.**

Let $P(n)$ be the statement 8 divides $3^{2n} - 1$. Then $P(0)$ is the statement 8 divides $3^{2 \times 0} - 1$. Since $3^{2 \times 0} - 1 = 1 - 1 = 0$ and 8 divides 0, $P(0)$ is true.

Next we assume that $P(k)$ is true for some integer $k \geq 0$, so 8 divides $3^{2k} - 1$. This means that $3^{2k} - 1 = 8m$ for some integer $m$. Let $f(k) = 3^{2k} - 1 = 8m$, then $3^{2k} = f(k) + 1$. Using these statements we want to prove that $P(k + 1)$ is true, that is, 8 divides $3^{2(k+1)} - 1$.

\[
3^{2(k+1)} - 1 = 3^{2k+2} - 1 = 3^{2k} \times 3^2 - 1 = 3^{2k} \times 9 - 1 = (f(k) + 1) \times 9 - 1 \text{ using } 3^{2k} = f(k) + 1 = 9 \times f(k) + 9 - 1 = 9 \times f(k) + 8 = 9 \times 8m + 8 \text{ using our inductive assumption} = 8(9m + 1)
\]

Since $9m + 1$ is an integer when $m$ is an integer, 8 divides $8(9m + 1)$ so 8 divides $3^{2(k+1)} - 1$ and $P(k+1)$ is true. Therefore, by mathematical induction, 8 divides $3^{2n} - 1$. 
