3 Differentiation

Differentiation is the main tool for finding the rate of change of a function at a given point. If a function represents displacement then we can use differentiation to find the rate of change of displacement, that is, velocity. If a function represents velocity then we can use differentiation to find the acceleration. If a function represents population size then we can find the rate of change of the population, that is, population growth or decline. Differentiation can also provide us with useful information for curve sketching, and is essential for solving optimisation problems.

3.1 Tangent Lines

To find the slope of a line \( y = f(x) = mx + c \) we evaluate \( \frac{f(b) - f(a)}{b - a} \), or ‘rise over run’. For a non-linear function \( y = f(x) \), the expression \( \frac{f(b) - f(a)}{b - a} \) does not give the slope of the function but as we decrease the distance between \( b \) and \( a \) we get a more accurate approximation of the slope of the tangent line at \( a \). If we evaluate the limit of the above expression as the difference between \( b \) and \( a \) goes to 0, we have the exact slope of the tangent line to \( y = f(x) \) at \( x = a \). We use the limit \( \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \) to do this.

**Example** Find the equation of the tangent line to \( f(x) = 3x^3 + x^2 - 4 \) at \( x = -3 \).

First we need to find the slope of the tangent line.

\[
\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{f(-3 + h) - f(-3)}{h} \\
= \lim_{h \to 0} \frac{(3(-3 + h)^3 + (-3 + h)^2 - 4) - (3(-3)^3 + (-3)^2 - 4)}{h} \\
= \lim_{h \to 0} \frac{3(-27 + 27h - 9h^2 + h^3) + 9 - 6h + h^2 - 4 + 81 - 9 + 4}{h} \\
= \lim_{h \to 0} \frac{-81 + 81h - 27h^2 + 3h^3 - 6h + h^2 + 81}{h} \\
= \lim_{h \to 0} \frac{3h^3 - 26h^2 + 75h}{h} \\
= \lim_{h \to 0} (3h^2 - 26h + 75) \\
= 75
\]

So the slope of the tangent at \( x = -3 \) is \( m = 75 \). Since the tangent is a straight line, we can use the formula \( m = \frac{y - y_1}{x - x_1} \). We know that the point \((-3, f(-3))\) is
on the line, that is, \((-3, -76)\).

\[
75 = \frac{y + 76}{x + 3}
\]

\[
75x + 225 = y + 76
\]

\[
y = 75x + 149
\]

### 3.2 The Derivative of a Function

The formula we used to calculate the slope of the tangent line in the previous section is also used to calculate the derivative of \(f\) at \(a\), \(f'(a)\). The derivative always represents the rate of change of the function. The greatest value of the derivative occurs where there is the steepest possible slope, and a horizontal line has derivative 0. We will introduce rules for quickly calculating derivatives in the next section but it is also possible to calculate derivatives using the definition; this is known as calculating from first principles.

**Example** Determine the derivative of \(f(x) = x^3\) from first principles.

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

\[
= \lim_{h \to 0} \frac{(x + h)^3 - x^3}{h}
\]

\[
= \lim_{h \to 0} \frac{x^3 + 3hx^2 + 3h^2x + h^3 - x^3}{h}
\]

\[
= \lim_{h \to 0} \frac{3hx^2 + 3h^2x + h^3}{h}
\]

\[
= \lim_{h \to 0} (3x^2 + 3hx + h^2)
\]

\[
= 3x^2
\]

As noted in Chapter 2 it is important to be able to sketch graphs. It is also important to be able to sketch the derivative of a function from a sketch of the function \(f\), and vice versa.

We commonly denote the derivative of \(y = f(x)\) by \(\frac{dy}{dx}\), which is read as the derivative of \(y\) with respect to \(x\). Note that the phrase ‘with respect to’ is often abbreviated as w.r.t.

### 3.3 Differentiation Rules

There are a number of rules that we can use for evaluating derivatives, some of them specific to one type of function, others more general. When they are stated in
general form, they can sometimes look more complicated than they are, so you may find it useful to look at some examples at the same time as you look at the general form. The rules can be used in combination to tackle more complicated functions.

The derivative of a function $f(x) = x^n$ is $f'(x) = nx^{n-1}$ for $n \in \mathbb{R}$. What is particularly useful here is that this works for all $n$ in the real numbers, not just integers, so for example, the derivative of $f(x) = x^{7/3}$ is $f'(x) = \frac{7}{3}x^{4/3}$.

The product rule and quotient rule are useful in evaluating many types of derivatives. They tend to be easier to remember in their shorthand version, and are often written using $u$ and $v$ to avoid confusion with functions labeled $f$. The product rule is $(uv)' = u'v + v'u$. The quotient rule is \( \left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2} \). Note that the order of $u'v$ and $v'u$ does not matter in the product rule since they are being summed, but the numerator of the quotient rule must start with $u'v$.

**Examples**

- **Find the derivative of** $g(x) = \frac{2x^3 + 1}{3x^2}$. Since this is a function involving division, we use the quotient rule. Let $u = 2x^3 + 1$ and $v = 3x^2$. So, using our rule for power functions, $u' = 6x^2$ and $v' = 6x$. Now we can use the quotient rule.

\[
g'(x) = \frac{u'v - v'u}{v^2} = \frac{6x^2 \times 3x^2 - 6x(2x^3 + 1)}{(3x^2)^2} = \frac{18x^4 - 12x^4 - 6x}{9x^4} = \frac{6x^4 - 6x}{9x^4} = \frac{3x(2x^3 - 2)}{3x(3x^3)} = \frac{2x^3 - 2}{3x^3}
\]

Notice that in this case we expanded the denominator and this allowed us to cancel a factor of $3x$ from the numerator and denominator. Since $v$ consisted of only a single term we simplified the $v^2$ in the denominator. However, as we shall see in the next example, in functions with a denominator consisting of more than one term we usually expand and simplify the numerator but do not expand the denominator.

- **Find the derivative of** $f(z) = \frac{3z^3 - 6z + 4}{2z^4 - 7}$. Again, since this is a function involving division, we use the quotient rule. Let $u(z) = 3z^3 - 6z + 4$ and
\[ v(z) = 2z^4 - 7. \] So, \( u'(z) = 9z^2 - 6 \) and \( v'(z) = 8z^3 \). We put these together using the quotient rule.

\[
f'(z) = \frac{u'v - v'u}{v^2}
= \frac{(9z^2 - 6)(2z^4 - 7) - 8z^3(3z^3 - 6z + 4)}{(2z^4 - 7)^2}
= \frac{18z^6 - 63z^2 - 12z^4 + 42 - 24z^6 + 48z^4 - 32z^3}{(2z^4 - 7)^2}
= \frac{-6z^6 + 36z^4 - 32z^3 - 63z^2 + 42}{(2z^4 - 7)^2}
\]

It is also possible to use the product rule and the chain rule instead of the quotient rule, using the power law \( a^{-m} = \frac{1}{a^m} \).

**Chain Rule** The chain rule is used to find the derivative of composite functions. One way to work out when to apply the chain rule is to look for a smaller expression inside a larger one. In these cases you find the derivative of the outside function (leaving the inner function as it is), and then multiply by the derivative of the inner function. So for two functions \( f(x) \) and \( g(x) \), \( (f \circ g)'(x) = f'(g(x))g'(x) \). We can also let \( y = f(u) \) and \( u = g(x) \) and rewrite the chain rule as \( \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \).

**Examples**

- **Find the derivative of** \( y = (3x^4 - 7x^2)^6 \). We can use the chain rule to find this derivative without having to expand 6 sets of brackets. Let \( u = 3x^4 - 7x^2 \), so \( y = u^6 \). Now we need to find the derivatives of these functions, \( \frac{du}{dx} = 6u^5 \), \( \frac{dy}{du} = 6u^5 \)

\[
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}
= 6u^5(12x^3 - 14x)
= 6(3x^4 - 7x^2)^5(12x^3 - 14x)
= 12x(3x^4 - 7x^2)^5(6x^2 - 7)
\]

- **Find the derivative of** \( y = \frac{7}{(3x^4 + \sqrt{x})^6} \). This function is a fraction so it could be evaluated with the quotient rule and the chain rule but it can be done more easily if we write it in the form, \( y = 7(3x^4 + x^{\frac{1}{2}})^{-6} \). Now using the chain rule, let \( u = 3x^4 + x^{\frac{1}{2}} \), the inner function, so \( y = 7u^{-6} \). Next we find
the two derivatives we need in order to apply the chain rule, \( \frac{dy}{du} = -42u^{-7} \), \( \frac{du}{dx} = 12x^3 + \frac{1}{2}x^{-\frac{1}{2}} \). Now applying the chain rule.

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \\
= -42u^{-7}(12x^3 + \frac{1}{2}x^{-\frac{1}{2}}) \\
= -42(3x^4 + x^{\frac{1}{2}})^{-7}(12x^3 + \frac{1}{2}x^{-\frac{1}{2}}) \\
= \frac{-42 \left(12x^3 + \frac{1}{2\sqrt{x}} \right)}{(3x^4 + \sqrt{x})^{7}}
\]

The final step was to put the derivative back in the same form as the original function, that is, as a fraction and with square root signs. We do not expand the denominator as this would involve expanding 7 sets of brackets.

**Implicit Differentiation** Implicit differentiation is the tool we use when we do not have a function in the form \( y = f(x) \). We know how to find the tangent line to the graph of a function \( y = f(x) \) at a given point, and we can also find the tangent line to the graph representing a relationship between \( x \) and \( y \) that cannot be written in the form \( y = f(x) \). For example, the graph below illustrates a relationship between \( x \) and \( y \) and it has a tangent line at every point \((x, y)\), such as the one shown at \( x = -2, y \approx 1.23 \).

![Graph of a function illustrating implicit differentiation](image)

We can use implicit differentiation to find the slope of the tangent line to a curve, at any point. To do this, you differentiate both sides of the equation with respect to whatever variable you need, say \( x \). We use the chain rule to differentiate the terms which are an expression of some other variable, say \( y \). For example
\[
\frac{d}{dx} 3y^2 = \frac{d}{dx} 3y^2 \frac{dy}{dx} = 6y \frac{dy}{dx}. \quad \text{The } \frac{dy}{dx} \text{ terms can then be collected, and the expression rearranged to give the derivative of } y \text{ with respect to } x.
\]

**Example** Find the derivative of \( y \) w.r.t. \( x \) where \( y \) is defined by the equation \( 3y^2 - y^3 + 1 - x + 3x^2 = 0 \).

\[
\frac{d}{dx} (3y^2 - y^3 + 1 - x + 3x^2) = \frac{d}{dx} (0)
\]
\[
\frac{d}{dx} 3y^2 - \frac{d}{dx} y^3 + \frac{d}{dx} 1 - \frac{d}{dx} x + \frac{d}{dx} 3x^2 = 0
\]
\[
\frac{d}{dx} 3y^2 - \frac{d}{dx} y^3 + 0 - 1 + 6x = 0
\]
\[
\frac{d}{dy} 3y^2 \frac{dy}{dx} - \frac{d}{dy} y^3 \frac{dy}{dx} + 6x - 1 = 0
\]
\[
6y \frac{dy}{dx} - 3y^2 \frac{dy}{dx} + 6x - 1 = 0
\]
\[
\frac{dy}{dx} (6y - 3y^2) = 1 - 6x
\]
\[
\frac{dy}{dx} = \frac{1 - 6x}{6y - 3y^2}
\]

As was mentioned above, this derivative could then be used to find the slope of the tangent line at any point \((x, y)\) that satisfies our equation \( 3y^2 - y^3 + 1 - x + 3x^2 = 0 \).

Note that we follow a similar process when we find the derivative of a function in the form we are more used to, say \( y = 3x^2 \).

\[
\frac{d}{dx} y = \frac{d}{dx} 3x^2
\]
\[
\frac{dy}{dx} \frac{dy}{dx} = 6x
\]
\[
1 \times \frac{dy}{dx} = 6x
\]
\[
\frac{dy}{dx} = 6x
\]

**Derivatives of Trigonometric, Exponential and Logarithmic Functions**

The derivatives of these functions are ones that you should memorise, although they can be found from first principles. They are important building blocks of larger functions. We can use the various techniques and simple derivatives from this section to find the derivatives of more complicated functions.

**Example** Find the derivative of \( f(x) = \ln(x^2 + 6x)e^{\cos x} \).
First we notice that this is the product of two functions, so we use the product rule. Let

\[ u = \ln(x^2 + 6), \quad v = e^{\cos x} \]

Now we need to find \(u'\) and \(v'\) and to do this we use the chain rule. So consider \(u = \ln(x^2 + 6)\), let \(m = x^2 + 6\), so \(u = \ln m\).

\[
\frac{dm}{dx} = 2x + 6 \\
\frac{du}{dm} = \frac{1}{m} \\
\frac{du}{dx} = \frac{du}{dm} \cdot \frac{dm}{dx} = \frac{1}{m}(2x + 6) = \frac{2x + 6}{x^2 + 6x}
\]

Now considering \(v = e^{\cos x}\), let \(n = \cos x\) so \(v = e^n\).

\[
\frac{dn}{dx} = -\sin x \\
\frac{dv}{dn} = e^n \\
\frac{dv}{dx} = \frac{dv}{dn} \cdot \frac{dn}{dx} = e^n(-\sin x) = -\sin x e^{\cos x}
\]

Therefore, applying the product rule,

\[
\frac{df}{dx} = u'v + v'u \\
= \frac{2x + 6}{x^2 + 6x} e^{\cos x} + (-\sin x) e^{\cos x} \ln(x^2 + 6x) \\
= e^{\cos x} \left( \frac{2x + 6}{x^2 + 6x} - (\sin x) \ln(x^2 + 6x) \right)
\]

**Critical Points and Curve Sketching** When we are sketching a function it is important to know where there are local maxima and local minima. These are examples of critical points. We identify critical points by solving \(f'(a) = 0\) for all possible values of \(a\) in our domain. Once we have identified the critical points, we classify them using the first or second derivative test.

To use the second derivative test we find the second derivative of the function (the derivative of the derivative) and evaluate it at the values \(a\) in which we are interested. If \(f''(a) > 0\) then we have a minimum; this can be remembered by thinking of a minimum as a smile, when the second derivative is positive we have a smile. If
\( f''(a) < 0 \) we have a maximum; similarly think of a negative second derivative forming a sad face, the shape of a maximum. If \( f''(a) = 0 \) then the second derivative test is inconclusive and we use the first derivative test to classify the critical point(s).

The second derivative test is generally easier to use if the function is not too complicated, but the first derivative test is also useful. To apply the first derivative test we look at the value of the first derivative at values near to our critical point \( a \). If the first derivative is positive at values slightly less than \( a \), and negative at values slightly larger than \( a \) then we have a local maximum. If the first derivative is negative at values slightly less than \( a \), and positive at values slightly larger than \( a \) then we have a local minimum. If there is no change of sign in the first derivative then we have neither a local maximum or a local minimum at \( a \). The ability to be able to find and classify critical points is necessary for accurate sketching of curves.

**Examples**

- **Find and classify the critical points of the function** \( f(x) = 3x^3 - 4x + 7 \) **using the first derivative test.**

  We have to solve \( f'(x) = 0 \) for all possible values of \( x \) to find the critical points.

  \[
  f'(x) = 9x^2 - 4 \quad \text{so let}
  
  9x^2 - 4 = 0
  
  9x^2 = 4
  
  x^2 = \frac{4}{9}
  
  x = \pm \frac{2}{3}
  \]

  Now we use the first derivative test to classify the critical points. First, we will classify the point at \( x = \frac{2}{3} \). We need to find the value of the first derivative at \( x \) values close to \( x = \frac{2}{3} \), so

  \[
  f'\left(\frac{1}{2}\right) = 9 \left(\frac{1}{2}\right)^2 - 4 = \frac{9}{4} - 4 = -\frac{7}{4} \quad \text{and}
  
  f'(1) = 9 \times 1^2 - 4 = 9 - 4 = 5.
  \]

  Since the first derivative changes from negative to positive we have a local minimum at \( x = \frac{2}{3} \). It is important to know the exact coordinates of the critical point, so we must also evaluate \( f\left(\frac{2}{3}\right) \).

  \[
  f\left(\frac{2}{3}\right) = 3 \left(\frac{2}{3}\right)^3 - 4 \left(\frac{2}{3}\right) + 7 = \frac{8}{9} - \frac{8}{3} + 7 = \frac{47}{9}
  \]

  We have a local minimum at \( \left(\frac{2}{3}, \frac{47}{9}\right) \). Next we classify the other critical point, at \( x = -\frac{2}{3} \), again using the first derivative test.

  \[
  f'(-1) = 9(-1)^2 - 4 = 9 - 4 = 5 \quad \text{and}
  \]
\[ f' \left( -\frac{1}{2} \right) = 9 \left( -\frac{1}{2} \right)^2 - 4 = \frac{9}{4} - 4 = -\frac{7}{4} \]

Since the first derivative changes from positive to negative we have a local maximum at \( x = -\frac{2}{3} \).

\[ f \left( -\frac{2}{3} \right) = 3 \left( -\frac{2}{3} \right)^3 - 4 \left( -\frac{2}{3} \right) + 7 = -\frac{8}{9} + \frac{8}{3} + 7 = \frac{79}{9} \]

We have a local maximum at \((-\frac{2}{3}, \frac{79}{9})\). This information can be used to sketch the graph of the function.

---

- Find and classify the critical points of the function \( f(x) = x^3 - x^2 - 16x + 7 \) using the second derivative test.

We have to solve \( f'(x) = 0 \) for all possible \( x \) values to find the critical points.

\[
\begin{align*}
    f''(x) &= 3x^2 - 2x - 16 = 0 \\
    (3x - 8)(x + 2) &= 0
\end{align*}
\]

So \( 3x - 8 = 0 \) or \( x + 2 = 0 \) which gives us \( x = -2 \) and \( x = \frac{8}{3} \). Now we need to use the second derivative, \( f''(x) = 6x - 2 \) to test the two critical points.

1. \( f''(-2) = -14 < 0 \) so we have a local maximum at \( x = -2 \), at the point \((-2, f(-2))\), which is \((-2, 27)\).
2. \( f''(\frac{8}{3}) = 14 > 0 \) so we have a local minimum at \( x = \frac{8}{3} \), at the point \((\frac{8}{3}, f(\frac{8}{3}))\), which is \((\frac{8}{3}, -\frac{643}{27})\).

Again, we can use this information to sketch the function.
3.4 Applications of Differentiation

When we are dealing with an optimisation problem involving multiple variables we are usually given several functions, some that give the relationship between two or more variables and one which we want to optimise. We substitute the functions which give the relationships between the variables into the function we want to optimise so that we are optimising a function in only one variable. We then take the derivative of the function we want to optimise and find its critical points, by finding when the derivative equals 0. We then use the first or second derivative test to check if the critical points we have are local maxima or minima, and if this information is suitable to answer the question. If boundary conditions have been imposed on the solution, we also need to check the boundary points, as each of them may be a global maximum or minimum. A final point to remember is that the solution we find should make sense in the context of the question; sometimes negative or non-integer values do not make sense. For example, if a question asked us to maximise the number of oranges that would fit inside a given box, the answer would need to be a positive integer.

**Examples**

- A farmer has 500m of fence to create a rectangular pen. Find the dimensions that maximise the area of this pen.

  Let $l =$ length and $w =$ width. We know $500 = 2l + 2w$ so

  \[
  250 = l + w \\
  l = 250 - w.
  \]

  We now want to find an expression for the area of the pen (which we are trying
to maximise) in terms of only one variable.

\[ A = lw \]
\[ = (250 - w)w \]
\[ = 250w - w^2 \]

Next, we find the derivative of \( A \) w.r.t. \( x \) and equate it to 0. The derivative is \( A' = 250 - 2w \) so let

\[ 250 - 2w = 0 \]
\[ 250 = 2w \]
\[ w = 125. \]

So we know that we have a critical point at \( w = 125 \), but we have to check that this is a maximum. We can do this using the second derivative test.

\[ A'' = -2 < 0 \] for all values of \( w \).

So we have a local maximum at \( w = 125 \). Since the equation for \( A \) in terms of \( w \) is a quadratic it has one turning point and so this local maximum is also a global maximum. The corresponding length for \( w = 125 \) is \( l = 250 - w = 125 \). The area of the pen will be maximised with dimensions 125 metres by 125 metres.

- **A closed rectangular box has a surface area of 2500 cm\(^3\) and the length of the base is to be three times the width of the base. Find the dimensions of the box that maximise its volume.**

Let \( l = \text{length}, w = \text{width} \) and \( h = \text{height} \). We know

\[ l = 3w \]  \hspace{1cm} (1)
\[ 2(lw + wh + hl) = 2500. \]  \hspace{1cm} (2)

So substituting (1) into (2) we can use the known surface area to obtain an expression for the height.

\[ 2(3w^2 + wh + 3wh) = 2500 \]
\[ 6w^2 + 8wh = 2500 \]
\[ 8wh = 2500 - 6w^2 \]
\[ h = \frac{2500 - 6w^2}{8w} \]

Now we can find an equation for volume in terms of just one variable, \( w \).

\[ V = lwh \]
\[ = (3w)(w)\left(\frac{2500 - 6w^2}{8w}\right) \]
\[ = \frac{7500w - 18w^3}{8} \]
We next find the critical points of the equation for the volume. The derivative of \( V \) is \( V' = \frac{7500 - 54w^2}{8} \) and we set \( V' = 0 \).

\[
\frac{7500 - 54w^2}{8} = 0 \\
7500 = 54w^2 \\
w^2 = \frac{7500}{54} \\
w \approx \pm 11.79
\]

Note that we only take the positive value since a negative width of a box does not make sense. We now have to check if this is a local maximum or a local minimum.

\[
V'' = -\frac{108w}{8} \\
V''(11.79) \approx -\frac{108(11.79)}{8} \\
V''(11.79) \approx -159.1 < 0
\]

So we know that at \( w \approx 11.79 \) we have a local maximum. To check the boundary conditions we consider the volume of the box when one of \( w, l \) or \( h \) is zero, and clearly this results in a box of volume zero, which is not a maximum. Therefore we have a global maximum at \( w \approx 11.79 \). Substituting this value in we can find the corresponding values of \( l \) and \( h \),

\[
l = 3w \approx 35.36 \\
h = \frac{2500 - 6w^2}{8w} \approx 17.68.
\]

The volume of the box will be maximised with approximately a width of 11.79cm, a length of 35.36cm, and a height of 17.68cm.