Factorising cubic polynomials.

$$f(x) = x^3 + bx^2 + cx + d$$
$$= (x-\alpha)(x-\beta)(x-\delta)$$
$$= x^3 \qquad - \alpha\beta\delta$$

$d = -\alpha\beta\delta$

$f(\alpha) = 0$

$$f(x) = x^3 + 2x^2 - 5x - 6$$

$f(0) = -6$, $f(1) = -8$, $f(2) = 0$

$$f(x) = (x-2)(ax^2 + bx + c)$$
$$= (x-2)(x^2 + bx + 3)$$

coeff of x^2 $\quad b - 2 = 2 \Rightarrow b = 4$

$$f(x) = (x-2)(x^2+4x+3)$$
$$= (x-2)(x+1)(x+3)$$

$2, -1, -3$

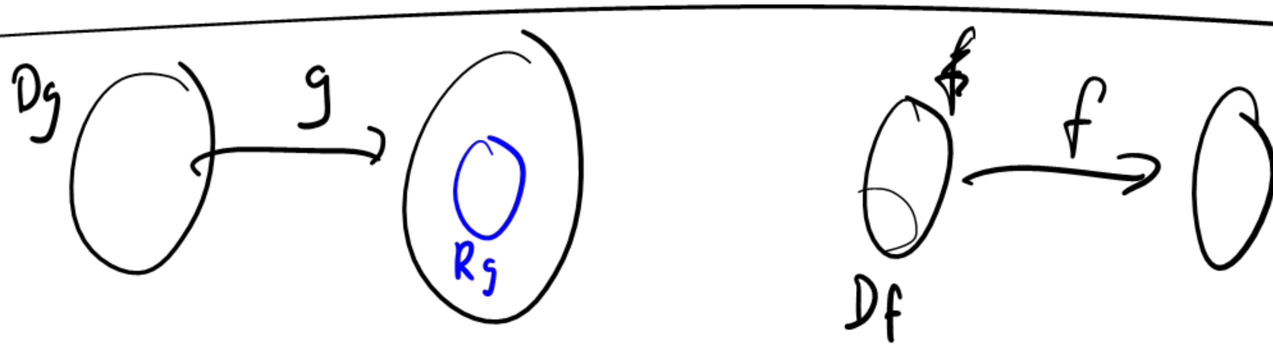
$$f(x) = 2x^3 + 3x^2 - 3x - 2$$

$f(0) = -2$, $f(1) = 2 + 3 - 3 - 2 = 0$

$$f(x) = (x-1)(ax^2 + bx + c)$$
$$= (x-1)(2x^2 + bx + 2)$$

Coeff x^2 : $-2+b = 3 \Rightarrow b = 5$

So $f(x) = (x-1)(2x^2+5x+2)$
$= (x-1)(2x+1)(x+2)$



$f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$
$g : [0,\infty) \to [0,\infty)$, $g(x) = \sqrt{x}$

$f\circ g(x) = f(\sqrt{x}) = (\sqrt{x})^2 = x$
$g\circ f(x) : g(x^2) = \sqrt{x^2} = x$

$g\circ f(x) = x$ $\qquad D_{g\circ f} \subseteq D_f = (-\infty,\infty)$

$f\circ g(x) = x$ $\qquad D_{f\circ g} \subseteq D_g = [0,\infty)$

$R_{f\circ g} : [0,\infty)$, $R_{g\circ f} : [0,\infty)$

2.4 Composition of functions

- Often we apply one function and then apply another function to the result of the first function. $\cdot \xrightarrow{g} \cdot \xrightarrow{f} \cdot$
- Given any two functions f and g, we start with a number x in the domain of g and determine its image $g(x)$. If this number is in the domain of f, we can now apply f to the number $g(x)$ and obtain the value $f(g(x))$. This process is called the *composition* of f and g. $x \to g(x) \to f(g(x))$
- Given two functions f and g, the function with input x and output $f(g(x))$ is called the *composite function* of f and g and is denoted $f \circ g$. $f \circ g\,(x)$
- The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.
- To apply the function $f \circ g$, we must first apply g and then apply f.

Example 2.4.1 Let $f(x) = x + 1, h(x) = 4x, i(x) = x^2$ and $j(x) = \sqrt{x}$.

Find (a) $f(h(-2))$ (b) $j(f(8))$ (c) $i(h(x))$ (d) $h(i(x))$

(a) $= f(-8) = -7$

(b) $= j(9) = 3$

(c) $= i(4x) = 16x^2$

(d) $= h(x^2) = 4x^2$

$i \circ h \neq h \circ i$

***Example** 2.4.2* Let f and g be functions defined by the formulae $f(x) = \sqrt{x-5}$ and $g(x) = x^2 - 4$. Determine the functions $f \circ g$ and $g \circ f$, and state their domains and ranges.

$D_f : [5, \infty)$, $D_g : (-\infty, \infty)$

$f \circ g(x) = f(g(x)) = f(x^2 - 4)$

$= \sqrt{(x^2-4) - 5} = \sqrt{x^2 - 9}$

$D_{f \circ g} : \quad x^2 \geq 9 \Rightarrow x \geq 3$ or $x \leq -3$

$D_{f \circ g} : (-\infty, -3] \cup [3, \infty)$

$R_{f \circ g} : [0, \infty)$

$g \circ f(x) = g(f(x)) = g(\sqrt{x-5})$

$= (\sqrt{x-5})^2 - 4$

$= x - 9$

$D_{g \circ f} \subseteq D_f$

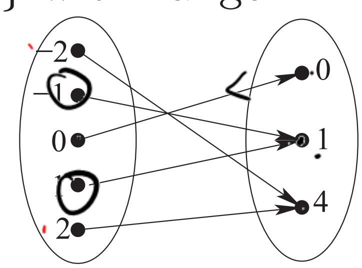
$D_{g \circ f} : [5, \infty)$

$R_{g \circ f} : [-4, \infty)$

2.5 Inverse functions

- If f is a function, then each element x in the domain of f gives rise to a unique value $f(x)$ in the range of f. However, it may be the case that multiple elements in the domain gives rise to the same element in the range.

- Let f be the function defined by $f(x) = x^2$, so $f(2) = 4$ and $f(-2) = 4$. The question: 'Which value of x gives rise to the value 4?' does not have a unique answer. Consider the arrow diagram illustrating f on the domain $A = \{-2, -1, 0, 1, 2\}$ with range $B = \{0, 1, 4\}$.



$f^{-1}(0) = 0$

$f^{-1}(1) = ?$

The value 4 has two arrows pointing to it, so if we try to reverse the action of f, there is confusion over what to do with 4. Hence, there is no function that reverses the action of f to map elements in B back to their corresponding elements in A.

$\log_{10}(x)$ ✓

- A function f is called a *one-to-one function* if it never takes the same value twice; that is

$f(x) = 3x - 1$ ✓

$$\text{if } x_1 \neq x_2 \Rightarrow \quad f(x_1) \neq f(x_2) \quad \text{for } x_1 \neq x_2.$$

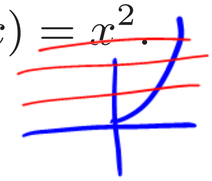
$f(x) = x^2$ ✗

- Let f be a one-to-one function with domain A and range B. Then its *inverse function* f^{-1} has domain B and range A and is defined by

$$f^{-1}(b) = a \text{ if and only if } f(a) = b$$

for any b in B.

***Example** 2.5.1* Let $f(x)$ be the function defined by $f(x) = x^2$. Determine a suitable domain for f so that f^{-1} exists.



$f_1 : [0, \infty) \to [0, \infty)$, $f_1(x) = x^2$

$f_1^{-1} = \sqrt{x}$

$f_2 : (-\infty, 0] \to [0, \infty)$, $f_2 = x^2$

$f_2^{-1} = -\sqrt{x}$

$y = x^2$

$x = \pm\sqrt{y}$

$f^{-1}(x) = \pm\sqrt{x}$

- To determine the inverse of a one-to-one function f:
  1. Solve the equation $y = f(x)$ for x in terms of y.
  2. To express f^{-1} as a function of x, interchange x and y in the result of the previous step.

***Example** 2.5.2* Determine the inverse function of $f(x) = 2x + 3$ and sketch the graphs of f and f^{-1} on the same set of axes.