Part A

For each of the following 18 multiple choice questions, enter the letter corresponding to the correct answer in the corresponding box. There is no need to show any working. Each correct answer is worth 1 mark; each incorrect answer is worth 0 marks.

1. Which of the following statements is false?
   (A) $-3 \in \mathbb{Z}$
   (B) $\mathbb{Q} \subseteq \mathbb{R}$
   (C) $\pi \in \mathbb{R}$
   (D) $\{-1, -2\} \cap \{1, 2\} = \phi$
   (E) $\{-1, -2\} \cup \{1, 2\} = \phi$
   (F) $\{4\} \subseteq \{4, 5, 3, 1, 2\}$

   Answer to Question 1: E

2. Simplify $\left(\frac{\frac{a}{b} + \frac{c}{d}}{\frac{a}{b} - \frac{c}{d}}\right)$.
   (A) $\frac{2(ad + bc)}{bd}$
   (B) $\frac{ad + bc}{ad - bc}$
   (C) $\frac{(ad)^2 - (bc)^2}{(bd)^2}$
   (D) $-1$
   (E) $0$
   (F) $1$

   Answer to Question 2: B

3. Let $f(x) = x^2 + 1$. Which of the following is the domain of $f(x)$?
   (A) $(-\infty, \infty)$
   (B) $(-\infty, 0]$
   (C) $[0, \infty)$
   (D) $(0, \infty)$
   (E) $(1, \infty)$
   (F) $[1, \infty)$

   Answer to Question 3: A

continued...
4. Let \( f(x) = x^2 + 1 \). Which of the following is the range of \( f(x) \)?

(A) \((-\infty, \infty)\)

(B) \((-\infty, 0]\)

(C) \([0, \infty)\)

(D) \((0, \infty)\)

(E) \((1, \infty)\)

(F) \([1, \infty)\)

Answer to Question 4: \(F\)

5. Which of the following is the range of the function \( f : [-2, 7] \rightarrow \mathbb{R}, f(x) = 5 - x \)?

(A) \([-2, 7]\)

(B) \((-2, 7]\)

(C) \((-2, \infty)\)

(D) \((-2, 7)\)

(E) \((-\infty, \infty)\)

(F) \([7, -2)\)

Answer to Question 5: \(B\)

6. If \( \sin \theta = \frac{\sqrt{3}}{2} \) and \( 0 \leq \theta < 2\pi \), then \( \theta \) equals which one of the following?

(A) \(\frac{\pi}{3}\)

(B) \(-\frac{\pi}{3}\)

(C) \(\frac{\pi}{6}\)

(D) \(-\frac{\pi}{6}\)

(E) \(\frac{\pi}{3} \) or \(\frac{5\pi}{6}\)

(F) \(\frac{\pi}{3} \) or \(\frac{2\pi}{3}\)

Answer to Question 6: \(F\)

continued...
7. If \( y = \log_x 1 \) where \( x \in \mathbb{R} \setminus \{0\} \), determine \( y \).

(A) 0
(B) 1
(C) 10
(D) \( e \)
(E) \( x \)
(F) None of the above

Answer to Question 7: A

8. If \( f(x) = 5x + 2 \), which of the following is the inverse function, \( f^{-1}(x) \)?

(A) \( f^{-1}(x) = 5x - 2 \)
(B) \( f^{-1}(x) = -5x - 2 \)
(C) \( f^{-1}(x) = \frac{1}{5}x + \frac{2}{5} \)
(D) \( f^{-1}(x) = \frac{1}{5}x - \frac{1}{2} \)
(E) \( f^{-1}(x) = \frac{1}{5}x - \frac{2}{5} \)
(F) \( f^{-1}(x) = -\frac{1}{5}x - \frac{2}{5} \)

Answer to Question 8: E

9. Determine \( \lim_{x \to 1} \frac{x^2 - 1}{x - 1} \).

(A) 0
(B) 1
(C) 2
(D) 3
(E) 4
(F) Does not exist

Answer to Question 9: C

continued...
10. Determine \( \lim_{x \to -3} (9 - x^2) \).

(A) \(-3\)
(B) 0
(C) 3
(D) 18
(E) 4
(F) Does not exist

Answer to Question 10: [B]

11. For the graph of \( y = f(x) \) shown below, \( f'(x) \) is negative over which interval?

(A) \(-3 < x < 3\)
(B) \(-3 \leq x \leq 3\)
(C) \(x < -3\) or \(x > 3\)
(D) \(x \leq -3\) or \(x \geq 3\)
(E) \(-5 < x < 1\) or \(x > 4\)
(F) \(-3 < x < \infty\)

Answer to Question 11: [A]

continued...
12. For the function $f(x)$ shown in the box, which of the following would be the best representation of $\frac{1}{f(x)}$?

Answer to Question 12: e

continued...
13. Which one of the following statements is true?
   (A) \( \sin(\pi + \theta) = \cos \theta \)
   (B) \( \sin^2 \theta - 1 = \cos^2 \theta \)
   (C) \( \tan(-\theta) = \frac{\sin(\theta)}{\cos(-\theta)} \)
   (D) \( \tan \theta \cos \theta = \sin \theta \)
   (E) \( \sin\left(\frac{3\pi}{2} - \theta\right) = \sin \theta \)
   (F) \( -\cos\left(\frac{\pi}{2} + \theta\right) = -\sin \theta \)

Answer to Question 13: [ ] D

14. What are the equations of the asymptotes of the graph of \( f(x) = \frac{x - 2}{x + 3} \)?
   (A) \( x = 3, y = 1 \)
   (B) \( x = 3, y = -2 \)
   (C) \( x = 3, y = 0 \)
   (D) \( x = -3, y = 1 \)
   (E) \( x = -3, y = \frac{2}{3} \)
   (F) \( x = -3, y = 2 \)

Answer to Question 14: [ ] D

15. If \( y = e^{\sin x} \), which one of the following statements is true?
   (A) \( \frac{dy}{dx} = e^{\sin x} \)
   (B) \( \frac{dy}{dx} = e^{\sin x - 1} \)
   (C) \( \frac{dy}{dx} = \cos x \cdot e^{\sin x} \)
   (D) \( \frac{dy}{dx} = \sin x \cdot e^{\sin x} \)
   (E) \( \frac{dy}{dx} = -\sin x \cdot e^{\sin x} \)
   (F) \( \frac{dy}{dx} = \cos x \cdot e^{\cos x} \)

Answer to Question 15: [ ] C

continued...
16. Which one of the following is NOT true about the graph of \( y = \log_e x \)?

(A) It has a vertical asymptote with equation \( x = 0 \).
(B) It passes through the point \((e, 0)\).
(C) The slope of the tangent at any point on the graph is positive.
(D) It has domain \((0, \infty)\).
(E) It has range \(\mathbb{R}\).
(F) It is the reflection in \( y = x \) of \( f(x) = e^x \).

Answer to Question 16: \[ \boxed{B} \]

17. Let \( p(x) = (x^2 + a)(x + b)(x - c) \), where \( a, b \) and \( c \) are three distinct positive real numbers. How many real solutions to the equation \( p(x) = 0 \) are there?

(A) 0
(B) 1
(C) 2
(D) 3
(E) 4
(F) 5

Answer to Question 17: \[ \boxed{C} \]

18. Which of the following intervals gives the solution to the inequality \( | -x| < 3 \)?

(A) \([-2, 2]\)
(B) \((-3, 3)\)
(C) \((-\infty, -3) \cup (3, \infty)\)
(D) \((-\infty, -3] \cup [3, \infty)\)
(E) \((-3, 3) \setminus \{0\}\)
(F) \([-3, 3]\)

Answer to Question 18: \[ \boxed{B} \]
19. Each graph in the second column is the graph of the derivative of one of the functions whose graphs are in the first column. Match each graph of the function with the graph of its derivative. (4 marks)

Answers to Question 19

Graph A: 3  Graph B: 4  Graph C: 2  Graph D: 1

End of Part A
1. Consider the function \( f(x) = (x - 1)^2 (x - 2) + 1 \). If \( f'(x) = (x - 1)(ax + b) \), where \( a \) and \( b \) are constants, use differentiation to find the values of \( a \) and \( b \).

\[
\text{Let } u = (x - 1)^2 \quad \nu = x - 2
\]
\[
u' = 2(x - 1) \quad \nu' = 1
\]
\[
f'(x) = (x - 1)^2 \cdot 1 + 2(x - 1)(x - 2) \quad \checkmark
\]
\[
= (x - 1)(x - 1 + 2x - 4)
\]
\[
= (x - 1)(3x - 5) \quad \checkmark
\]

\[
\therefore \quad a = 3 \quad b = -5 \quad \checkmark
\]
2. Given that \( f(-2) = -4, f(0) = 1, f'(-2) = 0, f'(x) < 0 \) when \( x < -2 \) and \( f'(x) > 0 \) when \( x > -2 \), sketch a possible graph of the function \( f(x) \). (4 marks)
3. Determine from first principles the derivative of \( f(x) = e^x \). \hfill (4 \text{ marks})

Note: \( f'(x) = \lim_{{h \to 0}} \frac{f(x+h) - f(x)}{h} \) and \( \lim_{{h \to 0}} \frac{e^h - 1}{h} = 1 \)

\[
f'(x) = \lim_{{h \to 0}} \frac{e^{x+h} - e^x}{h}
= \lim_{{h \to 0}} \frac{e^x \cdot e^h - e^x}{h}
= \lim_{{h \to 0}} \frac{e^x (e^h - 1)}{h}
= e^x \cdot \lim_{{h \to 0}} \frac{e^h - 1}{h}
= e^x \cdot 1
= e^x
\]

continued...
4. Let \( f(x) = \sqrt{-x+1} \) and \( g(x) = x^2 + 2 \).

a. Determine \( f(g(x)) \).
\[
f(g(x)) = \sqrt{-(x^2+2)+1} = \sqrt{-x^2-1}
\]

b. Determine \( g(f(x)) \).
\[
g(f(x)) = \left(\sqrt{-x+1}\right)^2 + 2 = -x + 1 + 2 = -x + 3
\]

(c. State the domain and range of \( g(f(x)) \). Explain your reasoning.

\[f: \text{Domain is } (-\infty, 1] \quad \checkmark\]
\[\text{Range is } [0, \infty) \]

\[g: \text{Domain is } \mathbb{R} \quad \checkmark\]
\[\text{Range is } [2, \infty) \]

\[g(f(x)): \text{Domain is } (-\infty, 1] \text{ as } f \text{ is only defined over } (-\infty, 1] \quad \checkmark\]
\[\text{Range therefore is } [2, \infty). \quad \checkmark\]
5. A model used for the number of tonnes of potatoes produced as a function of the level of nitrogen in the soil (in appropriate units) is \( P(N) = \frac{kN}{1 + N^2} \), where \( k \) is a positive constant. Find the level of nitrogen that will produce the most potatoes.

\[
\text{let } u = kN, \quad v = 1 + N^2 \\
u' = k, \quad v' = 2N
\]

\[
\therefore \quad p'(N) = \frac{k(1 - N^2) - kN \cdot 2N}{(1 + N^2)^2} = \frac{k - kN^2 - 2kN^2}{(1 + N^2)^2}
\]

\[
= \frac{k - kN^2}{(1 + N^2)^2}
\]

Critical points when \( p'(N) = 0 \)

\[
0 = \frac{k - kN^2}{(1 + N^2)^2}
\]

\[
0 = k - kN^2 \\
0 = k(1 - N^2) \\
0 = 1 - N^2 \\
N^2 = 1 \\
N = \pm 1
\]

However, can't have negative nitrogen, so \( N = 1 \).

Check max/min: \( p''(N) \) at \( p'(N) \)
6. a. Let \( f(x) = (\sin 2x - 3x^2)^3 \). Determine \( f'(x) \).

- Let \( u = \sin 2x - 3x^2 \) then \( f(u) = u^3 \)
- \( \frac{du}{dx} = 2\cos 2x - 6x \)
- \( \frac{df}{du} = 3u^2 \)

\[
f'(x) = (2\cos 2x - 6x) \cdot 3u^2 = 3(2\cos 2x - 6x)(\sin 2x - 3x^2)^2
\]

\( -\frac{1}{2} \) each mistake

b. Determine the derivative of \( y \) with respect to \( x \) if

\[\frac{2}{x} + \ln y = 2x^4 - 5y + 3.\]

\[-2x^{-2} + \frac{1}{y} \frac{dy}{dx} = 8x^3 - 5 \frac{dy}{dx}\]

\[-\frac{1}{y} \frac{dy}{dx} + 5 \frac{dy}{dx} = 8x^3 + \frac{2}{x^2}\]

\[\frac{dy}{dx} \left( \frac{1}{y} + 5 \right) = 8x^3 + \frac{2}{x^2}\]

\[
\frac{dy}{dx} = \frac{8x^3 + \frac{2}{x^2}}{\frac{1}{y} + 5}
\]

Continued...
7. For the graph of \( y = 4x^3 + 27x^2 - 30x + 10 \), determine the interval for which the gradient of the graph is negative. (6 marks)

\[
y' = 12x^2 + 54x - 30 \quad \checkmark
\]

Want
\[
12x^2 + 54x - 30 < 0 \quad \checkmark
\]

\[
6(2x^2 + 9x - 5) < 0
\]

\[
2x^2 + 9x - 5 < 0
\]

\[
(2x - 1)(x + 5) < 0 \quad \checkmark
\]

Either
\[
2x - 1 > 0 \quad \text{and} \quad x + 5 < 0
\]

\[
x > \frac{1}{2} \quad x < -5
\]

No interval

Or
\[
2x - 1 < 0 \quad \text{and} \quad x + 5 > 0
\]

\[
x < \frac{1}{2} \quad x > -5
\]

So
\[
\left( -5, \frac{1}{2} \right) \quad \checkmark
\]

Gradient is negative over \( \left( -5, \frac{1}{2} \right) \)

\[\text{continued...}\]
8. Find the maximum slope of the function $y = -x^3 + 3x^2 + 9x - 27$. (6 marks)

\[ y' = -3x^2 + 6x + 9 \checkmark \]
\[ y'' = -6x + 6 \checkmark \]

Max. slope is when the derivative of the slope is 0. So when $y'' = 0$

\[ -6x + 6 = 0 \checkmark \]
\[ 6x = 6 \]
\[ x = 1 \checkmark \]

\[ y''' \left( y^{(3)} \right) = -6 \checkmark \]

\[ \therefore \text{ Max. at } x = 1. \]

The slope at $x = 1$ is $-3(1)^2 + 6(1) + 9$

\[ = 12 \checkmark \]

End of Exam

formulae sheet... 6