MATH1050
Mathematical Foundations

Time: Two hours for working
Ten minutes for perusal before examination begins

CREDIT WILL BE GIVEN ONLY FOR WORK WRITTEN ON
THIS EXAMINATION SCRIPT.
Use the back pages if the space provided is insufficient.
Candidates should attempt ALL questions. All questions carry the indicated
number of marks. There are 60 marks in total. Check that this examination
paper has 23 printed pages.

FAMILY NAME (PRINT): ____________________________

GIVEN NAMES (PRINT): __________________________

STUDENT NUMBER: _____________________________

SIGNATURE: ________________________________
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TURN OVER
If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\det(A) = ad - bc$.

If $\det(A) \neq 0$, then $A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

If $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$, then $||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ and $\mathbf{\hat{v}} = \frac{1}{||\mathbf{v}||} \mathbf{v}$.

Scalar product: $\mathbf{v} \cdot \mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3$.

Vector product: $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$.

Cosine Law: $a^2 = b^2 + c^2 - 2bc \cos A$  Sine Law: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

Series: $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$, $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$.

The definition of the derivative $f'(a)$ of a function $f$ at the number $a$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$  

Differentiation rules:

$$(fg)' = f'g + fg', \quad \left( \frac{f}{g} \right)' = \frac{f'g - fg'}{g^2}, \quad (f \circ g)'(x) = f'(g(x))g'(x).$$

If $ax^2 + bx + c = 0$ then $x = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a}$.

Trigonometric identities:

$$\sin(-\theta) = -\sin(\theta)$$
$$\cos(-\theta) = \cos(\theta)$$
$$\sin(\theta + \phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi)$$
$$\cos(\theta + \phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)$$

Euler’s formula: $e^{i\theta} = \cos \theta + i \sin \theta$.

De Moivre’s theorem: If $z = r(\cos(\theta) + i \sin(\theta))$ then $z^n = r^n(\cos(n\theta) + i \sin(n\theta))$. 
For each of the following 10 multiple choice questions, write the letter corresponding to the correct answer in the space provided at the end of the question. Each correct answer is worth 0.5 marks, each incorrect answer is worth 0 marks.

1. Which of the following statements is true? ____
   (a) \( \sin \left( \theta + \frac{\pi}{2} \right) = \sin(\theta) \)
   (b) \( \sin \left( \theta + \frac{\pi}{2} \right) = \cos(\theta) \)
   (c) \( \sin \left( \theta + \frac{\pi}{2} \right) = \tan(\theta) \)
   (d) none of the above

2. If \( |x| < 2 \) which of the following statements is true? ____
   (a) \( 0 < x \leq 2 \)
   (b) \( -2 \leq x < 2 \)
   (c) \( -2 < x \leq 2 \)
   (d) \( -2 < x < 2 \)

3. Which of the following intervals corresponds to the inequality \( | -x | > 3 \)? ____
   (a) \( (-\infty, -3) \cup (3, \infty) \)
   (b) \( (-\infty, -3] \cup [3, \infty) \)
   (c) \( (-3, 3) \)
   (d) \( [-3, 3] \)

4. The domain of the function \( f(x) = a - x^2 \) is: ____
   (a) \( (-\infty, \infty) \)
   (b) \( (-a, \infty) \)
   (c) \( (0, \infty) \)
   (d) \( [-a, \infty) \)

5. The range of the function \( f(x) = a - x^2 \) is: ____
   (a) \( (-\infty, \infty) \)
   (b) \( (-\infty, a) \)
   (c) \( (0, \infty) \)
   (d) \( (-\infty, a] \)
6. Let \( u \), \( v \) and \( w \) be any vectors in \( \mathbb{R}^3 \). Which of the following statements is generally false? ____

(a) \( u \cdot v = v \cdot u \)
(b) \( u \times v = -v \times u \)
(c) \( (u \times v) \times w = u \times (v \times w) \)
(d) none of the above

7. If \( S = 2x - 4x^2 + 6x^3 - 8x^4 \), which of the following statements is true? ____

(a) \( S = 2 \sum_{k=0}^{3} (-1)^k (k+1)x^{k+1} \)
(b) \( S = 2 \sum_{k=1}^{4} (-1)^{k+1} (k+1)x^k \)
(c) \( S = 2 \sum_{k=0}^{3} (-1)^{(k+1)} (k+1)x^{k+1} \)
(d) none of the above

8. If \( A \) and \( B \) are invertible \( n \times n \) matrices and \( C = AB \), which of the following statements is true? ____

(a) \( C \) is singular
(b) \( C \) non-singular
(c) \( C \) is diagonal
(d) none of the above.

9. If \( y = x^3 + 2x^2 + 4x \) then \( \frac{dy}{dx} \) is: ____

(a) \( x^2 + 2x + 4 \)
(b) \( 3x^2 + 4x + 4 \)
(c) \( 3x + 4 \)
(d) \( 3x^2 + 4x \)

10. If \( y = x^3 + 2x^2 + 4x \) then \( \int y \, dx \) is: ____

(a) \( \frac{1}{4} x^4 + \frac{4}{3} x^3 + x^2 + c \)
(b) \( x^4 + \frac{4}{3} x^3 + 2x^2 + x \)
(c) \( x^4 + \frac{2}{3} x^3 + x^2 + x \)
(d) \( \frac{1}{4} x^4 + \frac{2}{3} x^3 + 2x^2 + c \)
11. Ticket prices for the Livid Festival are $90 full price and $70 for concessions. The total number of people attending is 35,000, and the takings from ticket sales are $2,884,960. Set up a pair of simultaneous equations and solve them using matrices in order to determine how many of each type of ticket were sold. (3 marks)
11. continued
12. A triangle has vertices in three-dimensional space given by the points $A = (0, 0, 0)$, 
$B = (2, 0, 3)$ and $C(0, 1, 0)$. Below, give all answers correct to two decimal places.

(a) Calculate $||\vec{AB}||$, $||\vec{BC}||$ and $||\vec{AC}||$. (2 marks)
(b) Calculate the angle at $A$. (1 mark)
(c) Using the vector cross product, find the area of the triangle. (2 marks)
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12. continued
13. Uschi and Hans go roller skating. Hans has a mass of 70 kg, and skates at a constant speed of 10 m/s. Uschi has a mass of 50 kg, is carrying a bag with mass 5 kg, and skates at a constant speed of 6 m/s. They collide at an angle of 140 degrees. During the collision, Hans accidently grabs Uschi’s bag. If Hans and the bag are at rest after the collision, use the conservation of momentum to determine the speed of Uschi immediately after the collision. (4 marks)
13. continued
14. (a) Recall that a closed formula for a geometric sequence is of the form \( a_k = ar^{k-1} \)
where \( a \) is the first term and \( r \) is the common ratio. Defining the series
\[
S_n = \sum_{k=1}^{n} a_k,
\]
prove by induction that
\[
S_n = \frac{a(r^n - 1)}{r - 1}.
\]
(3 marks)

(b) An exponential population model has the form \( P_n = P_0(1 + k)^n \) where \( k \) is the growth rate, \( P_0 \) is the initial population and \( P_n \) is the population after \( n \) years. If a particular species of cockroaches has an initial population of 10, and a population of 10,000 five years later, determine the growth rate \( k \) (give answer to two decimal places). (1 mark)
15. The first six terms of an infinite polynomial sequence \( \{a_n\}_{n=1}^{\infty} \) are given by

\[
a_1 = -1, \ a_2 = -2, \ a_3 = -1, \ a_4 = 2, \ a_5 = 7, \ a_6 = 14.
\]

(a) Use the method of finite differences to determine a closed form for the sequence. (4 marks)

(b) From the result of part (a), determine a recursive formula for the sequence. (1 mark)
16. For this question, give all answers with rational denominators.

(a) Find the exact value of $\cos\left(\frac{5\pi}{12}\right)$ (where the angle is measured in radians). (2 marks)

(b) Find the exact value of $\cos\left(\frac{5\pi}{24}\right)$. (2 marks)
16. continued
17. Solve the following inequality. Write your solution in interval notation and illustrate it on a number line.

\[ \frac{|y - 4|}{y + 2} < 3. \]

(4 marks)
17. continued
18. Let \( f(x) = \sqrt{2x + 1} \) and let \( g(x) = \frac{1}{x^2} \).

(a) State the domain and range of \( f \). (1 mark)

(b) State the domain and range of \( g \). (1 mark)

(c) Determine \((g \circ f)(x)\) and state its domain and range. (2 marks)

(d) Is \( g \) a one-to-one function? Explain your answer. (1 mark)
18. continued
19. Let \( f(x) = \frac{3}{x - 2} \).

(a) Use the definition of the derivative as a limit to determine the slope of the tangent line at \( x = -1 \). (3 marks)

(b) Find the equation of the tangent line to \( f \) at \( x = -1 \). (2 marks)
19. continued
20. (a) Let \( z = \sqrt{3} - i \). Use De Moivre’s Theorem to evaluate \( z^3 \) and express your answer in polar form. (2 marks)

(b) One solution of \( 2z^3 - 9z^2 + 30z - 13 = 0 \) is \( z = 2 - 3i \). Find all the other solutions of this equation over \( \mathbb{C} \). (2 marks)

(c) Determine all values of \( z \in \mathbb{C} \) such that \( z^3 = -27i \). Illustrate your values of \( z \) on an Argand diagram. (3 marks)
20. continued
21. Determine the derivative for the following function.

(a) \( g(u) = 4u^2 e^{2u} \). (2 marks)

Evaluate the following indefinite integral.

(b) \( \int x^2 \sqrt{2x^3 + 4} \, dx \) (2 marks)
21. continued
22. Let \( f(x) = -3x^2 + 2x \).

(a) Evaluate the Riemann sum for \( f(x) \), with \( 0 \leq x \leq 2 \), having 20 subintervals and taking the sample points to be the right endpoints. (3 marks)

(b) Evaluate the definite integral \( \int_0^2 f(x) \, dx \). (2 marks)
22. continued